

# EQUATIONS OF HYPERELLIPTIC SHIMURA CURVES

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**ABSTRACT.** By constructing suitable Borchers forms on Shimura curves and using Schofer's formula for norms of values of Borchers forms at CM-points, we determine all the equations of hyperelliptic Shimura curves  $X_0^D(N)$ . As a byproduct, we also address the problem of whether a modular form on Shimura curves  $X_0^D(N)/W_{D,N}$  with a divisor supported on CM-divisors can be realized as a Borchers form, where  $X_0^D(N)/W_{D,N}$  denotes the quotient of  $X_0^D(N)$  by all the Atkin-Lehner involutions. The construction of Borchers forms is done by solving certain integer programming problems.

## 1. INTRODUCTION

For an indefinite quaternion algebra  $B$  of discriminant  $D$  over  $\mathbb{Q}$  and a positive integer  $N$  with  $(D, N) = 1$ , we let  $X_0^D(N)$  be the Shimura curve associated to an Eichler order  $\mathcal{O}$  of level  $N$  in  $B$ . When  $D = 1$ , the Shimura curve  $X_0^D(N)$  is simply the classical modular curve  $X_0(N)$ , which is the coarse moduli space of elliptic curves together with a cyclic subgroup of order  $N$  and has been extensively studied in literature. When  $D > 1$ , the curve  $X_0^D(N)$  is the coarse moduli space of principally polarized abelian surfaces with multiplication by  $\mathcal{O}$ . The arithmetic of such a Shimura curve is similar to those of classical modular curves, but the lack of cusps makes the Diophantine geometry and explicit calculation of such a Shimura curve more interesting and challenging than those of classical modular curves. The primary purpose of the present paper is to address the problem of determining equations of Shimura curves.

In the classical modular case, which have been extensively studied and well-known for admitting Fourier expansions around the cusps, there are many constructions of modular forms and modular functions, such as Eisenstein series, the Dedekind function, theta series, and etc., and there are formulas for their Fourier expansions. Thus, it is often easy to determine equations of modular curves. We refer the reader to Galbraith [14], Yang [30], and the references contained therein for more informations about equations of modular curves.

On the other hand, when  $D \neq 1$ , the absence of cusps has been an obstacle for explicit approaches to Shimura curves since modular forms or modular functions on Shimura curves do not have Fourier expansions and as a result, most of the methods for classical modular curves cannot possibly be extended to the case of general Shimura curves. Up to now, only a few equations of Shimura curves are known. Ihara [20] was perhaps the first one to give defining equations of Shimura curves. For example, he found an equation

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for the curve  $X_0^6(1)$  of genus 0. Kurihara [24] extended Ihara's method and determined equations of  $X_0^{10}(1)$  and  $X_0^{22}(1)$  of genus 0 and  $X_0^{14}(1)$ ,  $X_0^{21}(1)$ , and  $X_0^{46}(1)$  of genus 1. Jordan [21] computed equations of two Shimura curves  $X_0^{15}(1)$  and  $X_0^{33}(1)$  of genus 1. Later on, González and Rotger [16, 17] completed the list of equations of Shimura curves  $X_0^D(N)$  of genus 1 and 2. For Shimura curves  $X_0^D(N)$  of higher genus, Elkies [12] found equations of Shimura curves  $X_0^{57}(1)$  and  $X_0^{206}(1)$  using the fact that some families of  $K3$  surfaces are parameterized by Shimura curves. More recently, Molina [25] found equations of  $X_0^{39}(1)$  and  $X_0^{55}(1)$  and Atkin-Lehner quotients of some Shimura curves. Also, González and Molina [15] determine equations of all Shimura curves  $X_0^D(1)$  of genus 3. (Note that it happens that all these curves are hyperelliptic.) We remark that all the methods in the above-mentioned works other than those in [12] are strongly based on the Cerednik-Drinfeld theory of  $p$ -adic uniformization of Shimura curves [6],  $p|D$ , and arithmetic properties of CM-points. In addition, other than [12], their methods do not allow us to locate general CM-points on the curves.

In this paper, we will adopt a very different approach, using the theory of Borcherds forms and explicit formulas for values of Borcherds forms at CM-points to obtain equations of Shimura curves. (See Section 2 for a quick introduction to Borcherds forms.) The main result of this paper is a complete list of equations of all hyperelliptic Shimura curves  $X_0^D(N)$ .

**Theorem 1.** *The table in Appendix A gives a complete list of equations of hyperelliptic Shimura curves  $X_0^D(N)$ ,  $D > 1$ .*

The idea of realizing modular forms on Shimura curves as Borcherds forms is not new. For example, as a corollary to his formula for average values of Borcherds forms at CM-points, Schofer [27] proved a weak analogue of Gross and Zagier's result [18] on the prime factorization of the norm of the difference of two singular moduli on the classical modular curve  $X_0(1)$  for the case of Shimura curves. Later on, Errthum [13] applied Schofer's formula to compute singular moduli on  $X_0^6(1)/W_{6,1}$  and  $X_0^{10}(1)/W_{10,1}$ , verifying Elkies numerical computation [11], where  $W_{D,N}$  denotes the full Atkin-Lehner group on  $X_0^D(N)$ . However, applications of Borcherds forms to theory of Shimura curves were not explored any further in literature. One possible reason is that in order to successfully use Borcherds forms to do computation on Shimura curves, one needs a systematic method to construct them in the first place, but such a method is not developed in literature yet. Thus, our first task here is to develop a systematic method to construct Borcherds forms. We will see that the problem of constructing Borcherds forms reduces to that of solving certain integer programming problems, which we solve by using the AMPL modeling language (<http://www.ampl.com>) and the Gurobi solver (<http://www.gurobi.com>).

Note that our method works for any Shimura curve  $X_0^D(N)$  such that  $X_0^D(N)/W_{D,N}$  has genus 0, but because there are too many of them, here we consider only the hyperelliptic cases. (There are more than 110 non-hyperelliptic Shimura curves  $X_0^D(N)$  whose Atkin-Lehner quotient  $X_0^D(N)/W_{D,N}$  has genus 0.) In addition, under a certain technical assumption (Assumption below), it is also possible to determine equations of  $X_0^D(N)/W_{D,N}$  even if it is not of genus 0. In Section 4.3, we give two such examples. However, the method is less systematic and it is not clear whether it will always work in general.

In principle, our list of equations should also be obtainable using Elkies' approach [12], but our approach via Borcherds forms have potential applications to other problems about Shimura curves beyond the scope of the present paper. To illustrate our point, in Section 5, we will show how our construction of Borcherds forms lead to a method to compute

heights of CM-points on Shimura curves, again, under Assumption 40. Note that both Elkies' and our approaches have an advantage over other methods in that we can determine the coordinates of CM-points on Shimura curves.

The rest of the paper is organized as follows. In Section 2, we give a quick overview of the theory of Borchers forms and explain the idea of realizing modular forms on Shimura curves in terms of Borchers forms. The exposition of this section follows [31]. In Section 3, we discuss how to construct Borchers forms by solving certain integer programming problems. For our purpose, the case of odd  $D$  needs special attention. As a byproduct, we find that for  $(D, N)$  in Theorem 1 with even  $D$ , all meromorphic modular forms with divisors supported on CM-divisors (Definition 21) can be realized as Borchers forms. (We believe that this is also true for odd  $D$ , but since it is not the main problem we are concerned with, we will not prove this assertion here.) In Section 4, we will give several examples illustrating how to obtain equations of Shimura curves using Borchers forms we constructed in Section 3 and Schofer's formula for values of Borchers forms at CM-points. In Section 4.3, we give additional examples where the genus of  $X_0^D(N)/W_{D,N}$  is not 0. Specifically, we determine equations of  $X_0^{142}(1)/W_{142,1}$  and  $X_0^{302}(1)/W_{302,1}$ , under Assumption 40. Finally, in Section 5 we demonstrate how to explicitly compute heights of CM-points on Shimura curves using our construction of Borchers forms.

## 2. BORCHERS FORMS

**2.1. Basic theory.** We give a quick introduction to Borchers forms. For details, see [3, 4, 8] for the classical setting, see [13, 23, 27] for the adelic setting.

Let  $L$  be an even lattice with symmetric bilinear form  $\langle \cdot, \cdot \rangle$  of signature  $(n, 2)$  and  $L^\vee$  be the dual lattice of  $L$ . We assume  $L$  is nondegenerate and denote by

$$\{e_\eta : \eta \in L^\vee/L\}$$

the standard basis for the group algebra  $\mathbb{C}[L^\vee/L]$ . Associated to the lattice  $L$ , we have a unitary *Weil representation*  $\rho_L$  of the metaplectic group

$$\widetilde{\mathrm{SL}}(2, \mathbb{Z}) = \left\{ \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \sqrt{c\tau + d} \right) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}) \right\}$$

on the group algebra  $\mathbb{C}[L^\vee/L]$  defined by

$$\begin{aligned} \rho_L(T)e_\eta &= e^{-2\pi i \langle \eta, \eta \rangle / 2} e_\eta, \\ \rho_L(S)e_\eta &= \frac{e^{2\pi i (n-2)/8}}{\sqrt{|L^\vee/L|}} \sum_{\delta \in L^\vee/L} e^{2\pi i \langle \eta, \delta \rangle} e_\delta, \end{aligned}$$

where

$$S = \left( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{\tau} \right) \quad \text{and} \quad T = \left( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1 \right),$$

which generate  $\widetilde{\mathrm{SL}}(2, \mathbb{Z})$ .

**Definition 1.** A holomorphic function  $F : \mathfrak{H} \rightarrow \mathbb{C}[L^\vee/L]$  is called a weakly holomorphic vector-valued modular form of weight  $k \in \frac{1}{2}\mathbb{Z}$  and type  $\rho_L$  on  $\widetilde{\mathrm{SL}}(2, \mathbb{Z})$  if it satisfies

$$F\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k \rho\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \sqrt{c\tau + d}\right) F(\tau)$$

for all  $\tau \in \mathfrak{H}$  and all  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$  and  $F$  is meromorphic at the cusp  $\infty$ . The last condition means that the Fourier expansion of  $F$  is of the form

$$F(\tau) = \sum_{\eta \in L^\vee / L} \sum_{m \in \mathbb{Z} + \langle \eta, \eta \rangle / 2, m > -m_0} c_\eta(m) q^m e_\eta, \quad q = e^{2\pi\tau},$$

for some rational number  $m_0$ .

For  $k = \mathbb{Q}, \mathbb{R}$  or  $\mathbb{C}$ , let  $V(k) = L \otimes k$  and extend the definition of  $\langle \cdot, \cdot \rangle$  to  $V(k)$  by linearity. Define  $O_V(\mathbb{R})$  to be the orthogonal group of the bilinear form  $\langle \cdot, \cdot \rangle$  and its subgroup

$$O_V^+(\mathbb{R}) := \{\sigma \in O_V(\mathbb{R}) : \text{spin } \sigma = \text{sgn det } \sigma\},$$

where if  $\sigma$  is equal to the product of  $n$  reflections with respect to the vectors  $v_1, \dots, v_n$ , then its spinor norm is defined by  $\text{spin } \sigma = (-1)^n \prod_{i=1}^n \text{sgn } \langle v_i, v_i \rangle$ . We also define

$$O_L^+ := \{\sigma \in O_V^+(\mathbb{R}) : \sigma(L) = L\}$$

to be the orthogonal group of the lattice  $L$ . As the orthogonal group  $O_L^+$  acts on the dual lattice  $L^\vee$ , there is an induced operation on  $\mathbb{C}[L^\vee / L]$  given by

$$\sum_{\eta \in L^\vee / L} c_\eta e_\eta \mapsto \sum_{\eta \in L^\vee / L} c_\eta e_{\sigma\eta}, \quad \sigma \in O_L^+.$$

**Definition 2.** Suppose that  $F = \sum_{\eta \in L^\vee / L} F_\eta e_\eta$  is a vector-valued modular form. We define the automorphism group  $O_{L,F}^+$  of  $F$  by

$$O_{L,F}^+ = \{\sigma \in O_L^+ : F_{\sigma\eta} = F_\eta \text{ for all } \eta \text{ in } L^\vee / L\}.$$

Consider the subset

$$K = \{[z] \in \mathbb{P}(V(\mathbb{C})) : \langle z, z \rangle = 0, \langle z, \bar{z} \rangle < 0\}$$

of the projective space  $\mathbb{P}(V(\mathbb{C}))$ . This set  $K$  consists of two connected components and the orthogonal group  $O_V^+(\mathbb{R})$  preserves the components. Pick one of them to be  $K^+$ . Then it can be checked that  $O_V^+(\mathbb{R})$  acts transitively on  $K^+$ .

**Definition 3.** Suppose  $\tilde{K}^+ = \{z \in V(\mathbb{C}) : [z] \in K^+\}$ . For each subgroup of  $\Gamma$  of finite index of  $O_L^+$ , we call a meromorphic function  $\Psi : \tilde{K}^+ \rightarrow \mathbb{P}(\mathbb{C})$  a modular form of weight  $k$  and character  $\chi$  on  $\Gamma$  if  $\Psi$  satisfies

- (i)  $\Psi(cz) = c^{-k} \tilde{\Psi}(z)$  for all  $c \in \mathbb{C}^*$  and  $z \in \tilde{K}$ ,
- (ii)  $\Psi(hz) = \chi(h) \tilde{\Psi}(z)$  for all  $h \in \Gamma$  and  $z \in \tilde{K}$ .

**Theorem A** ([3, Theorem 13.3]). *Let  $L$  be an even lattice of signature  $(n, 2)$  and  $F(\tau)$  be a weakly holomorphic vector-valued modular forms of weight  $1 - n/2$  and type  $\rho_L$  with Fourier expansion  $F(\tau) = \sum_{\eta} (\sum_n c_\eta(n) q^n) e_\eta$ . Suppose that  $c_\eta(n) \in \mathbb{Z}$  for any  $\eta \in L^\vee / L$  and  $n \leq 0$ . Then there corresponds a meromorphic function  $\Psi_F(z)$ ,  $z \in \tilde{K}^+$  with the following properties.*

- (i)  $\Psi_F(z)$  is a meromorphic modular forms of weight  $c_0(0)/2$  for the group  $O_{L,F}^+$  with respect to some unitary character  $\chi$  of  $O_{L,F}^+$ .
- (ii) The only zeros or poles of  $\Psi_F(z)$  lie on the rational quadratic divisor

$$\lambda^\perp = \{z \in \tilde{K}^+ : \langle z, \lambda \rangle = 0\}$$

for  $\lambda$  in  $L$ ,  $\langle \lambda, \lambda \rangle > 0$ , and are of order

$$\sum_{0 < r \in \mathbb{Q}, r\lambda \in L^\vee} c_{r\lambda} (-r^2 \langle \lambda, \lambda \rangle / 2).$$

**Definition 4.** We call the function  $\Psi_F(z)$  the *Borcherds form* associated to  $F$ .

**2.2. Borcherds forms on Shimura curves.** We now explain how to realize modular forms on Shimura curves as Borcherds forms. We follow the exposition in [31]. See also [13].

Let  $B$  be an indefinite quaternion algebra of discriminant  $D$  over  $\mathbb{Q}$ . Consider the vector space

$$V = V(\mathbb{Q}) = \{x \in B : \text{tr}(x) = 0\}$$

over  $\mathbb{Q}$  with the natural bilinear form  $\langle x, y \rangle = \text{tr}(x\bar{y}) = -\text{tr}(xy)$ . Then  $V$  has signature  $(1, 2)$  and the associated quadratic form is  $\text{nm}(x) = -x^2$ . Given an Eichler order  $\mathcal{O}$  of level  $N$  in  $B$ , we let  $L$  be the lattice

$$L = \mathcal{O} \cap V = \{x \in \mathcal{O} : \text{tr}(x) = 0\}.$$

For an invertible element  $\beta$  in  $B \otimes \mathbb{R}$ , define  $\sigma_\beta : V(\mathbb{R}) \rightarrow V(\mathbb{R})$  by  $\sigma_\beta(\gamma) = \beta\gamma\beta^{-1}$ . Then, we can show that

$$O_V^+(\mathbb{R}) = \{\sigma_\beta : \beta \in (B \otimes \mathbb{R})^* / \mathbb{R}^*, \text{nm}(\beta) > 0\} \times \{\pm 1\}$$

and

$$O_L^+ = \{\sigma_\beta : \beta \in N_B^+(\mathcal{O}) / \mathbb{Q}^*\} \times \{\pm 1\}.$$

If we assume that the quaternion algebra is represented by  $B = (\frac{a,b}{\mathbb{Q}})$  with  $a > 0$  and  $b > 0$ , that is,  $B = \mathbb{Q} + \mathbb{Q}i + \mathbb{Q}j + \mathbb{Q}ij$  with  $i^2 = a$ ,  $j^2 = b$ , and  $ij = -ji$ , and fix an embedding  $\iota : B \hookrightarrow M(2, \mathbb{R})$  by

$$\iota : i \mapsto \begin{pmatrix} 0 & \sqrt{a} \\ \sqrt{a} & 0 \end{pmatrix}, \quad j \mapsto \begin{pmatrix} \sqrt{b} & 0 \\ 0 & -\sqrt{b} \end{pmatrix},$$

then each class in  $K = \{z \in \mathbb{P}(V(\mathbb{C})) : \langle z, z \rangle = 0, \langle z, \bar{z} \rangle < 0\}$  contains a unique representative of the form

$$z(\tau) = \frac{1 - \tau^2}{2\sqrt{a}}i + \frac{\tau}{\sqrt{b}}j + \frac{1 + \tau^2}{2\sqrt{ab}}ij$$

for some  $\tau \in \mathfrak{H}^\pm$ , the union of upper and lower half-plane. The mapping  $\tau \mapsto z(\tau) \bmod \mathbb{C}^*$  is a bijection of between  $\mathfrak{H}^\pm$  and  $K$ .

Let  $K^+$  be the image of  $\mathfrak{H}^+ = \mathfrak{H}$  under the mapping. Then we get compatible actions of  $N_B^+(\mathcal{O}) / \mathbb{Q}^*$  on  $K^*$  and  $\mathfrak{H}$  with the action on  $K^+$  by conjugation and the action on  $\mathfrak{H}$  by linear fraction transformation. More precisely, this means that for  $\alpha \in N_B^+(\mathcal{O})$ , if we write  $\iota(\alpha) = \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix}$ , then

$$(1) \quad \alpha z(\tau) \alpha^{-1} = \frac{(c_3\tau + c_4)^2}{\text{nm}(\alpha)} z \left( \frac{c_1\tau + c_2}{c_3\tau + c_4} \right) \equiv z(\iota(\alpha)\tau) \bmod \mathbb{C}^*.$$

**Lemma 5** ([31, Lemma 4]). *Let  $F(\tau) = \sum_\eta (\sum_n c_\eta(n) q^n) e_\eta$  be a weakly holomorphic vector-valued modular form of weight  $1/2$  and type  $\rho_L$  such that  $O_{L,F}^+ = O_L^+$  and  $c_\eta(n) \in \mathbb{Z}$  whenever  $\eta \in L^\vee / L$  and  $n \leq 0$ . Then the function  $\psi_F(\tau)$  defined by  $\psi_F(\tau) = \Psi_F(z(\tau))$  is a meromorphic modular forms of weight  $c_0(0)$  with certain unitary character  $\chi$  on the Shimura curve  $X_0^D(N) / W_{D,N}$ .*

**Definition 6.** With assumptions given as in the lemma, the function  $\psi_F(\tau)$  defined by

$$\psi_F(\tau) = \Psi_F(z(\tau))$$

is called the *Borcherds forms* on the Shimura curve  $X_0^D(N)/W_{D,N}$  associated to  $F$ .

The next lemma gives us the criterion when the character of a Borcherds form  $\psi_F(\tau)$  is trivial, under the assumption that the genus of  $N_B^+(\mathcal{O}) \backslash \mathfrak{H}$  is zero.

**Lemma 7** ([31, Lemma 6]). *Assume that the genus of  $X = N_B^+(\mathcal{O}) \backslash \mathfrak{H}$  is zero. Let  $\tau_1, \dots, \tau_r$  be the elliptic points of  $X$  and assume that their orders are  $b_1, \dots, b_r$ , respectively. Assume further that, as CM-points, the discriminant of  $\tau_1, \dots, \tau_r$  are  $d_1, \dots, d_r$ , respectively. Let  $F(\tau) = \sum_{\eta} (\sum_m c_{\eta}(n) q^m) e_{\eta}$  be a weakly holomorphic vector-valued modular form of weight  $1/2$  and type  $\rho_L$  such that  $O_{L,F}^+ = O_L^+$  and  $c_{\eta}(m) \in \mathbb{Z}$  whenever  $\eta \in L^{\vee}/L$  and  $m \leq 0$ . Assume that  $c_0(0)$  is even. Then the Borcherds form  $\psi_F(\tau)$  is a modular form with trivial character on  $X$  if and only if for  $j$  such that  $b_j \neq 3$ , the order of  $\Psi_F(z)$  at  $z(\tau_j)$  has the same parity as  $c_0(0)/2$ .*

We now state Schofer's formula [27, Corollaries 1.2 and 3.5] in the setting of Shimura curves as follows.

**Theorem B** ([27, Corollaries 1.2 and 3.5]). *Let  $F(\tau) = \sum_{\eta} (\sum_m c_{\eta}(n) q^m) e_{\eta}$  be a weakly holomorphic vector-valued modular form of weight  $1/2$  and type  $\rho_L$  for  $\widehat{\mathrm{SL}}(2, \mathbb{Z})$  such that  $O_{L,F}^+ = O_L^+$ ,  $c_0(0) = 0$ , and  $c_{\eta}(m) \in \mathbb{Z}$  whenever  $\eta \in L^{\vee}/L$  and  $n \leq 0$ . Let  $d < 0$  be a fundamental discriminant such that the set  $\mathrm{CM}(d)$  of CM-points of discriminant  $d$  on  $N_B^+(\mathcal{O}) \backslash \mathfrak{H}$  is not empty and that the support of  $\mathrm{div} \psi(\tau)$  does not intersect  $\mathrm{CM}(d)$ . Then we have*

$$\sum_{\tau \in \mathrm{CM}(d)} \log |\psi_F(\tau)| = -\frac{|\mathrm{CM}(d)|}{4} \sum_{\gamma \in L^{\vee}/L} \sum_{m \geq 0} c_{\gamma}(-m) \kappa_{\gamma}(m),$$

where  $\kappa_{\gamma}(m)$  are certain sums involving derivatives of Fourier coefficients of some incoherent Eisenstein series.

We refer the reader to [13, 31] for strategies to compute  $\kappa_{\gamma}(m)$  explicitly.

### 3. CONSTRUCTION OF BORCHERDS FORMS

**3.1. Errthum's method.** In this section, we will review Errthum's method [13] for constructing vector-valued modular forms out of scalar-valued modular forms. Here the notations  $D$ ,  $N$ ,  $\mathcal{O}$ ,  $L$ , and etc. have the same meanings as those in Section 2.2. The level  $N$  is always assumed to be squarefree.

Let us first describe the structure of the lattice  $L$ .

**Lemma 8.** *Assume that  $N$  is squarefree. Let  $q$  be a prime number such that  $q \equiv 1 \pmod{4}$  and*

$$(2) \quad \left( \frac{q}{p} \right) = \begin{cases} -1, & \text{if } p|D, \\ 1, & \text{if } p|N. \end{cases}$$

Then  $B = \left( \frac{DN \cdot q}{\mathbb{Q}} \right)$  is a quaternion algebra of discriminant  $D$  over  $\mathbb{Q}$ . Moreover, let  $a$  be an integer such that  $a^2 DN \equiv 1 \pmod{q}$ . Then the  $\mathbb{Z}$ -module  $\mathcal{O}$  generated by

$$(3) \quad e_1 = 1, \quad e_2 = \frac{1+j}{2}, \quad e_3 = \frac{i+j}{2}, \quad e_4 = \frac{aDNj+i}{q}$$

is an Eichler order of level  $N$  in  $B$ . Also, let  $L$  be the set of elements of trace zero in  $\mathcal{O}$  and let

$$(4) \quad \ell_1 = j, \quad \ell_2 = \frac{i + ij}{2}, \quad \ell_3 = \frac{aDNj + ij}{q}.$$

Then

$$L = \mathbb{Z}\ell_1 + \mathbb{Z}\ell_2 + \mathbb{Z}\ell_3, \quad L^\vee = \mathbb{Z}\frac{\ell_1}{2} + \mathbb{Z}\frac{\ell_2}{DN} + \mathbb{Z}\frac{\ell_3}{DN}.$$

*Proof.* The conditions in (2) imply that  $B$  is ramified at prime divisors of  $D$  and unramified at prime divisors of  $N$ . Also, by the quadratic reciprocity law, we have  $\left(\frac{DN}{q}\right) = 1$ . Thus, the discriminant of  $B$  is  $D$ .

We check that

$$\begin{aligned} e_2^2 &= \frac{q-1}{4}e_1 + e_2, \\ e_2e_3 &= \frac{aDN(q-1)}{4}e_1 + \frac{aDN(1-q)}{2}e_2 + \frac{1-q}{2}e_3 + \frac{q(q-1)}{4}e_4, \\ e_2e_4 &= aDNe_1 - aDNe_2 - e_3 + \frac{q+1}{2}e_4, \\ e_3^2 &= \frac{DN(1-q)}{4}e_1, \\ e_3e_4 &= -\frac{DN(a^2DN(q-1) + q + 1)}{2q}e_1 + \frac{DN(a^2DN(q-1) + 1)}{q}e_2 \\ &\quad + aDNe_3 + \frac{aDN(1-q)}{2}e_4, \end{aligned}$$

so that  $\mathbb{Z}e_1 + \mathbb{Z}e_2 + \mathbb{Z}e_3 + \mathbb{Z}e_4$  is an order in  $B$ . Also, the Gram matrix

$$(\text{tr}(e_i \bar{e}_j)) = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & (q-1)/2 & 0 & -aDN \\ 0 & 0 & DN(q-1)/2 & DN \\ 0 & -aDN & DN & 2DN(1-a^2DN)/q \end{pmatrix}$$

has determinant  $(DN)^2$ . Thus, it is an Eichler order of level  $N$ .

Moreover, it is clear that  $\ell_1, \ell_2$ , and  $\ell_3$  span  $L$ . Also, the Gram matrix of  $L$  with respect to this basis is

$$(5) \quad \begin{pmatrix} -2q & 0 & -2aDN \\ 0 & DN(q-1)/2 & DN \\ -2aDN & DN & 2DN(1-a^2DN)/q \end{pmatrix},$$

and its determinant is  $2D^2N^2$ . From the Gram matrix of  $L$ , it is easy to check that  $L^\vee$  is spanned by  $\ell_1/2, \ell_2/DN$  and  $\ell_3/DN$ . This proves the lemma.  $\square$

**Corollary 9.** Assume that  $N$  is squarefree. The discriminant of the lattice  $L$  is

$$|L^\vee/L| = 2(DN)^2$$

and the level of  $L$  is

$$\begin{cases} 4DN, & \text{if } DN \text{ is odd,} \\ 2DN, & \text{if } DN \text{ is even.} \end{cases}$$

*Proof.* The result follows directly from the proof in above lemma since the determinant of the Gram matrix in (5) is  $2(DN)^2$  and  $L^\vee/L \simeq (\mathbb{Z}/2) \times (\mathbb{Z}/DN)^2$ .  $\square$

We now recall Errthum's method [13] for constructing weakly holomorphic vector-valued modular forms. Let  $\chi_\theta$  denote the character associated to the Jacobi theta function  $\theta(\tau) = \sum_{n \in \mathbb{Z}} q^{n^2}$ . That is,  $\chi_\theta$  is defined by

$$\theta(\gamma\tau) = \chi_\theta(\gamma)(c\tau + d)^{1/2}\theta(\tau)$$

for all  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$  and all  $\tau \in \mathfrak{H}$ .

**Lemma 10** ([2, Theorem 4.2.9]). *Let  $M$  be the level of the lattice  $L$ . Suppose that  $f(\tau)$  is a weakly holomorphic scalar-valued modular form of weight  $1/2$  such that*

$$f(\gamma\tau) = \chi_\theta(\gamma)(c\tau + d)^{1/2}f(\tau)$$

*for all  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(M)$ . Then the function  $F_f(\tau)$  defined by*

$$(6) \quad F_f(\tau) = \sum_{\gamma \in \widetilde{\Gamma}_0(M) \setminus \widetilde{\text{SL}}(2, \mathbb{Z})} f|_\gamma(\tau) \rho_L(\gamma^{-1}) e_0$$

*is a weakly holomorphic vector-valued modular form of weight  $1/2$  and type  $\rho_L$ .*

**Lemma 11** ([13, Theorem 5.8]). *Let  $f(\tau)$  and  $F_f(\tau)$  be given as in previous lemma. Then for  $\eta$  and  $\eta' \in L^\vee/L$  with  $\langle \eta, \eta \rangle = \langle \eta', \eta' \rangle$ , the  $e_\eta$  component and  $e_{\eta'}$  component of  $F_f(\tau)$  are equal. Consequently, we have  $O_{L, F_f}^+ = O_L^+$ .*

**Lemma 12** ([4, Theorem 6.2]). *Let  $M$  be the level of the lattice  $L$ . Suppose that  $r_d, d|M$ , are integers satisfying the conditions*

- (i)  $\sum_{d|M} r_d = 1$ ,
- (ii)  $|L^\vee/L| \prod_{d|M} d^{r_d}$  is a square in  $\mathbb{Q}^*$ ,
- (iii)  $\sum_{d|M} dr_d \equiv 0 \pmod{24}$ , and
- (iv)  $\sum_{d|M} (M/d)r_d \equiv 0 \pmod{24}$ .

*Then  $\prod_{d|M} \eta(d\tau)^{r_d}$  is a weakly holomorphic scalar-valued modular form satisfying the condition for  $f(\tau)$  in Lemma 10.*

**Definition 13.** If an eta-product satisfies the conditions in Lemma 12, then we say it is *admissible*.

To have a better control over the divisors of Borchers forms constructed, we will use certain special admissible eta products.

**Definition 14.** Let  $M$  be the level of the lattice  $L$  and let  $S$  be a subset of the cusps of  $\Gamma_0(M)$ . If  $f$  is a weakly holomorphic modular form of weight  $1/2$  on  $\Gamma_0(M)$  whose only poles are at the cusps in  $S$ , then we say  $f$  is a  $S$ -weakly holomorphic scalar-valued modular form of weight  $1/2$  on  $\Gamma_0(M)$ .

Later on, we will use  $\{\infty\}$ -weakly holomorphic modular forms to construct Borchers forms for the case of even  $D$  and  $\{\infty, 0\}$ -weakly holomorphic modular forms for the case of odd  $D$ . Therefore, let us introduce the following definitions.

**Definition 15.** Let  $D_0$  be the odd part of  $DN$ . We let  $M^!(4D_0)$  denote the space of all  $\{\infty\}$ -weakly holomorphic modular forms of weight  $1/2$  on  $\Gamma_0(4D_0)$ . Also, for a nonnegative integer  $n$ , let  $M_n^!(4D_0)$  be the subspace of  $M^!(4D_0)$  consisting of modular forms with a pole of order  $\leq n$  at  $\infty$ . If  $j$  is a positive integer such that there does not exist a modular form in  $M^!(4D_0)$  with a pole of order  $j$  at  $\infty$ , then we say  $j$  is a *gap* of  $M^!(4D_0)$ .

Similarly, we let  $M^{!,!}(4D_0)$  be the space of all  $\{\infty, 0\}$ -weakly holomorphic modular forms of weight  $1/2$  on  $\Gamma_0(4D_0)$ . For nonnegative integers  $m$  and  $n$ , let  $M_{m,n}^{!,!}(4D_0)$  be



the subspace of  $M^{1,1}(4D_0)$  consisting of modular forms with a pole of order  $\leq m$  at  $\infty$  and a pole of order  $\leq n$  at 0.

**Remark 16.** Notice that the space  $M_0^1(4D_0)$  is simply the space of holomorphic modular forms of weight  $1/2$  on  $\Gamma_0(4D_0)$ . Since  $D_0$  is assumed to be squarefree, by Theorem A of [28], the space  $M_0^1(4D_0)$  is one-dimensional and spanned by  $\theta(\tau)$ .

**3.2. Case of even  $D$ .** In this section, we assume that  $D$  is even and  $N$  is squarefree. In Proposition 18, we will see how the problem of constructing Borchers forms becomes the problem of solving certain integer programming problem. Ultimately, in Proposition 23, we will show that for  $(D, N)$  in Theorem 1 with  $2|D$ , every meromorphic modular form of even weight on  $X_0^D(N)/W_{D,N}$  with divisor supported on CM-divisors (see Definition 21) can be realized as a Borchers form. Note that Bruinier [9] and Heim and Murase [19] studied when a modular form on an orthogonal group  $O(n, 2)$  can be realized as a Borchers form, but as the integer  $n$  is assumed to be at least 2, their results do not apply to the case of Shimura curves. In fact, it is pointed out in [9, Section 1] that counterexamples exist in the case  $n = 1$  (see also [7, Section 8.3]). It will be a very interesting problem to characterize those modular forms on Shimura curves  $X_0^D(N)/W_{D,N}$  that can be realized as Borchers forms.

Let  $D_0$  be the odd part of  $DN$ . Then according to Corollary 9, the level of the lattice under consideration is  $4D_0$ . Let us first determine the dimensions of  $M_n^1(4D_0)$ .

**Lemma 17.** *Let  $D_0$  be the odd part of  $DN$  and  $g$  be the genus of the modular curve  $X_0(4D_0)$ . Then for a nonnegative integer  $n$  with*

$$n \geq 2g - 2 - \sum_{d|D_0} \lfloor d/4 \rfloor,$$

*we have*

$$\dim_{\mathbb{C}} M_n^1(4D_0) = n + \sum_{d|D_0} \lfloor d/4 \rfloor + 1 - g.$$

*Moreover, the number of gaps of  $M^1(4D_0)$  is  $g - \sum_{d|D_0} \lfloor d/4 \rfloor$ .*

*Proof.* Let  $\theta(\tau) = \sum_n q^{n^2}$  be the Jacobi theta function. For a divisor  $d$  of  $4D_0$ , let  $C_d$  represent the cusp  $1/d$ . As a modular form on  $\Gamma_0(4D_0)$ , we have

$$\operatorname{div} \theta = \sum_{d|D_0} \frac{d}{4} (C_{2d}).$$

A modular form  $f$  is contained in  $M_n^1(4D_0)$  if and only if the modular function  $g = f/\theta$  on  $\Gamma_0(4D_0)$  satisfies

$$\operatorname{div} g \geq -n(\infty) - \sum_{d|D_0} \lfloor d/4 \rfloor (C_{2d}).$$

Then by the Riemann-Roch theorem, when  $n$  is a nonnegative integer such that  $n \geq 2g - 2 - \sum_{d|D_0} \lfloor d/4 \rfloor$ , the dimension of the space  $M_n^1(4D_0)$  is

$$n + \sum_{d|D_0} \lfloor d/4 \rfloor + 1 - g.$$

Now since  $D_0$  is squarefree, by Theorem A of [28], the space  $M_0^1(4D_0)$  is one-dimensional and spanned by  $\theta$ , which implies that there is no modular form in  $M^1(4D_0)$  having a zero at  $\infty$ . Therefore, from the dimension formula for  $M_n^1(4D_0)$ , we see that the number of gaps is  $n + 1 - \dim M_n^1(4D_0) = g - \sum_{d|D_0} \lfloor d/4 \rfloor$ .  $\square$

**Proposition 18.** *For  $(D, N)$  in Theorem 1 with even  $D$ , the space  $M^!(4D_0)$  is spanned by admissible eta-products. Moreover, there exists a positive integer  $m$  such that, for each positive integer  $j \geq m$ , there exists a modular form  $f_j$  in  $M^!(4D_0) \cap \mathbb{Z}((q))$  whose order of pole at  $\infty$  is  $j$  and whose leading coefficient is 1.*

*Proof.* Let  $g$  be the genus of the modular curve  $X_0(4D_0)$  and set

$$n_0 = \max(2g - 2 - \sum_{d|D_0} \lfloor d/4 \rfloor, 0).$$

According to Lemma 17, if  $n$  is an integer such that  $n \geq n_0$ , then there exists a modular form in  $M^!(4D_0)$  with a pole of order  $n$  at  $\infty$ . Now suppose that we can find an eta-product  $t(\tau)$  such that  $t$  is a modular function on  $\Gamma_0(4D_0)$  with a unique pole at  $\infty$ . Let  $k$  be the order of the pole of  $t$  at  $\infty$ . Now Lemma 17 implies that for each integer  $j \geq n_0$ , there is a modular form in  $M^!(4D_0)$  with a pole of order  $j$  at  $\infty$ . Thus, for all  $n \geq n_0$ , we have

$$M_{n+k}^!(4D_0) = M_n^!(4D_0) + tM_n^!(4D_0).$$

Therefore, to prove the assertion about  $M^!(4D_0)$ , it suffices to find such a modular function  $t$  and show that the space  $M_{n_0+k}^!(4D_0)$  can be spanned by eta-products and that there exists a positive integer  $m \geq n_0$  such that for each integer  $j$  with  $m \leq j \leq m+k-1$ , there exists a modular form in  $M_j^!(4D_0) \cap \mathbb{Z}((q))$  whose order of pole at  $\infty$  is  $j$  and whose leading coefficient is 1.

Consider the case of a maximal order first. Assume that  $D = 2p$  for some odd prime  $p$ . By Lemma 12, for an eta-product  $\prod_{d|4p} \eta(d\tau)^{r_d}$  to be admissible, the integers  $r_d$  must satisfy

$$(7) \quad \begin{array}{rclclclcl} r_1 & + & r_2 & + & r_4 & + & r_p & + & r_{2p} & + & r_{4p} & = & 1 \\ & & r_2 & & & & & + & r_{2p} & & & = & 1 + 2\delta_2 \\ & & & & & & r_p & + & r_{2p} & + & r_{4p} & = & 2\delta_p \\ r_1 & + & 2r_2 & + & 4r_4 & + & pr_p & + & 2pr_{2p} & + & 4pr_{4p} & = & 24\epsilon_1 \\ 4pr_1 & + & 2pr_2 & + & pr_4 & + & 4r_p & + & 2r_{2p} & + & r_{4p} & = & 24\epsilon_2 \end{array}$$

for some integers  $\delta_2$ ,  $\delta_p$ ,  $\epsilon_1$ , and  $\epsilon_2$ . Moreover, the congruence subgroup  $\Gamma_0(4p)$  has 6 cusps, represented by  $1/c$  with  $c|4p$ . The orders of the eta function  $\eta(d\tau)$  at these cusps, multiplied by 24, are given by

	1	1/2	1/4	1/p	1/2p	1/4p
$\eta(\tau)$	$4p$	$p$	$p$	$4$	$1$	$1$
$\eta(2\tau)$	$2p$	$2p$	$2p$	$2$	$2$	$2$
$\eta(4\tau)$	$p$	$p$	$4p$	$1$	$1$	$4$
$\eta(p\tau)$	$4$	$1$	$1$	$4p$	$p$	$p$
$\eta(2p\tau)$	$2$	$2$	$2$	$2p$	$2p$	$2p$
$\eta(4p\tau)$	$1$	$1$	$4$	$p$	$p$	$4p$

Thus, in order for an eta-product to be in  $M_n^!(4p)$ , the exponents  $r_d$  should satisfy

$$(8) \quad \begin{array}{rcccccccl} r_1 & + & 2r_2 & + & 4r_4 & + & pr_p & + & 2pr_{2p} & + & 4pr_{4p} & \geq & -24n \\ r_1 & + & 2r_2 & + & r_4 & + & pr_p & + & 2pr_{2p} & + & pr_{4p} & \geq & 0 \\ 4r_1 & + & 2r_2 & + & r_4 & + & 4pr_p & + & 2pr_{2p} & + & pr_{4p} & \geq & 0 \\ pr_1 & + & 2pr_2 & + & 4pr_4 & + & r_p & + & 2r_{2p} & + & 4r_{4p} & \geq & 0 \\ pr_1 & + & 2pr_2 & + & pr_4 & + & r_p & + & 2r_{2p} & + & r_{4p} & \geq & 0 \\ 4pr_1 & + & 2pr_2 & + & pr_4 & + & 4r_p & + & 2r_{2p} & + & r_{4p} & \geq & 0 \end{array}$$

In literature, problems of solving a set of equalities and inequalities in integers are called *integer programming problems*. Solving (7) and (8) using the AMPL modeling language (<http://www.ampl.com>) and the gurobi solver (<http://www.gurobi.com>), we can produce many admissible eta-products.

To find  $t$ , we replace the first two equations in (7) by  $r_1 + r_2 + r_4 + r_p + r_{2p} + r_{4p} = 0$  and  $r_2 + r_{2p} = 2\delta_2$  and solve the integer programming problem. We find that we can choose

$$t(\tau) = \frac{\eta(4\tau)^4 \eta(2p\tau)^2}{\eta(2\tau)^2 \eta(4p\tau)^4}$$

with  $k = (p - 1)/2$ .

In the other cases when  $N > 1$ ,  $N$  is always a prime. The modular curve  $X_0(4D_0)$  has 12 cusps and there are more inequalities and equalities in the integer programming problem. Nevertheless, we can easily find  $t$  and many admissible eta-products by solving the integer programming problem.

Having found  $t(\tau)$  and many admissible eta-products, we check case by case that eta-products do span  $M_{n_0+k}^!(4p)$  and that there exists positive integer  $m \geq n_0$  such that for each integer  $j$  with  $m \leq j \leq m + k - 1$ , there exists a modular form  $f_j \in M_{n_0+k}^!(4D_0) \cap \mathbb{Z}((q))$  whose order of pole at  $\infty$  is  $j$  and whose leading coefficient is 1. (Sometimes,  $f_j$  will be a linear combination of eta-products with rational coefficients. To show that all Fourier coefficients are integers, we use Sturm's theorem.) Here we omit the details, providing only one example as below.  $\square$

**Example 19.** Consider the case  $D = 26$  and  $N = 1$ . The modular curve  $X_0(52)$  has genus 5. Thus, by Lemma 17, the number of gaps of  $M^!(52)$  is  $5 - \sum_{d|13} \lfloor d/4 \rfloor = 2$ . The modular function

$$t(\tau) = \frac{\eta(4\tau)^4 \eta(26\tau)^2}{\eta(2\tau)^2 \eta(52\tau)^4}$$

has a unique pole of order 6 at  $\infty$ . According to the proof of Proposition 18, we need to show that the space  $M_{11}^!(52)$  can be spanned by eta-products. Using the gurobi solver, we find the following solutions  $(r_1, r_2, r_4, r_{13}, r_{26}, r_{52})$  to the integer programming problem in (7) and (8) with  $n = 11$

$$\begin{aligned} &(-3, 6, 0, -3, 11, -10), (-1, 3, 1, 3, 2, -7), (3, -3, 3, -1, 8, -9), (1, 1, 1, 1, 4, -7), \\ &(-1, 1, 1, 3, 4, -7), (0, -3, 6, -2, 8, -8), (3, -1, 1, -1, 6, -7), (-5, 12, -4, -1, 5, -6), \\ &(-1, 2, 0, -5, 15, -10), (1, -1, 1, 1, 6, -7), (1, 3, -1, 1, 2, -5), (-1, 3, -1, 3, 2, -5), \\ &(3, 1, -1, -1, 4, -5), (-2, 3, 2, 0, 2, -4), (0, -1, 2, -2, 6, -4), (-2, 5, -2, 0, 0, 0). \end{aligned}$$

Suitable linear combinations of these eta-products  $\prod_{d|52} \eta(d\tau)^{r_d}$  yield a basis consisting of

$$\begin{aligned} f_0 &= 1 + 2q + 2q^4 + 2q^9 + \cdots, & f_3 &= q^{-3} + q^{-1} + q^3 + q^9 + \cdots \\ f_4 &= q^{-4} - q^{-1} - q + q^3 + \cdots, & f_5 &= q^{-5} + q^{-2} - 2q + q^2 + \cdots, \\ f_6 &= q^{-6} + q^{-2} - 2q + 2q^2 + \cdots, & f_7 &= q^{-7} - q^{-2} + 2q - q^2 + \cdots, \\ f_8 &= q^{-8} + q^{-2} + q^2 + 2q^5 + \cdots, & f_9 &= q^{-9} + 2q^{-1} + 3q + 2q^3 + \cdots, \\ f_{10} &= q^{-10} + 3q^{-1} + q - q^3 + \cdots, & f_{11} &= q^{-11} + 2q^2 + q^5 + 4q^7 + \cdots, \end{aligned}$$

for the space  $M_{11}^1(52)$ . In fact, since all these modular forms have integral coefficients, multiplying these  $f_j$  by powers of  $t$ , we find that for each non-gap positive integer  $j$ , there exists a modular form  $f_j$  in  $M^1(52) \cap \mathbb{Z}((q))$  with a pole of order  $j$  at  $\infty$  and a leading coefficient 1.

**Remark 20.** Quite curiously, our computation shows that whenever  $N = 1$ , i.e., whenever  $D_0 = p$  is an odd prime, the space  $M^1(4D_0)$  has the property that for each non-gap positive integer  $j$ , there exists a modular form  $f$  in  $M^1(4D_0) \cap \mathbb{Z}((q))$  such that  $f$  has a pole of order  $j$  at  $\infty$  with leading coefficient 1.

The smallest  $D_0$  such that  $M^1(4D_0)$  does not have this property is  $D_0 = 51$ . We can show that the gaps of  $M^1(204)$  are  $1, \dots, 14$ , and 20 and there exists a modular form  $f$  in  $M^1(204) \cap \mathbb{Z}((q))$  with a Fourier expansion  $2q^{-22} - q^{-20} - 2q^{-14} + 2q^{-12} + \dots$ . As 20 is a gap, there cannot exist  $g \in M^1(204) \cap \mathbb{Z}((q))$  with a Fourier expansion  $q^{-22} + \dots$ .

We now show that for  $(D, N)$  in Theorem 1 with even  $D$ , all meromorphic modular forms of even weights on  $X_0^D(N)/W_{D,N}$  with divisors supported on CM-divisors, which we define below, can be realized as Borcherds forms.

**Definition 21.** For a negative discriminant  $d$ , we let  $\text{CM}(d)$  denote the set of CM-points of discriminant  $d$  on  $X_0^D(N)/W_{D,N}$ ,  $h_d = |\text{CM}(d)|$ , and  $P_d$  be the divisor

$$P_d = \sum_{\tau \in \text{CM}(d)} \tau.$$

(If  $h_d = 0$ , then  $P_d$  simply means 0.) We call  $P_d$  the CM-divisor of discriminant  $d$ . Note that sometimes we wish to keep track the degree of the divisor  $P_d$ . In such a case, we will write  $P_d^{\times h_d}$  instead of  $P_d$ .

**Lemma 22.** Let  $f$  be an element in  $M^1(4D_0) \cap \mathbb{Z}((q))$ ,  $F_f$  be the vector-valued modular form constructed using  $f$  as given by (6), and  $\psi_{F_f}(\tau)$  be the Borcherds form on  $X_0^D(N)/W_{D,N}$  corresponding to  $F_f$  as defined in Definition 6. Suppose that the Fourier expansion of  $f$  is  $\sum_m c_m q^m$ . Then

$$\text{div } \psi_{F_f} = \sum_{m < 0} c_m \sum_{r \in \mathbb{Z}^+, 4m/r^2 \text{ is a discriminant}} \frac{1}{e_{4m/r^2}} P_{4m/r^2},$$

where  $e_d$  is the cardinality of the stabilizer subgroup of  $\tau \in \text{CM}(d)$  in  $N_B^+(\mathcal{O})/\mathbb{Q}^*$ .

*Proof.* This follows from Proposition 5.4 of [13] and Lemma 7 of [31].  $\square$

**Proposition 23.** For  $(D, N)$  in Theorem 1 with  $2|D$ , all meromorphic modular forms of even weights on  $X_0^D(N)/W_{D,N}$  with a divisor supported on CM-divisors can be realized as Borcherds forms.

*Proof.* We will prove only the case  $(D, N) = (26, 1)$ . The proof of the other cases is similar.

We claim that

- (i) there is a Borchers form  $\psi$  of weight 2 with a trivial character, and
- (ii) every modular function on  $X_0^{26}(1)/W_{26,1}$  with divisor supported on CM-divisors can be realized as a Borchers form.

Then observe that if  $\phi$  is a modular form of even weight  $k$ , then  $\phi/\psi^{k/2}$  has weight 0. The two claims imply that  $\phi$  can be realized as a Borchers form.

The Shimura curve  $X_0^{26}(1)/W_{26,1}$  has genus 0 and precisely five elliptic points of order 2. Among the five elliptic points, one is a CM-point of discriminant  $-8$ , one is a CM-point of discriminant  $-52$ , and the remaining three are CM-points of discriminant  $-104$ . Also, if  $\psi$  is a meromorphic modular form of even weight  $k$  on  $X_0^{26}(1)/W_{26,1}$ , then the degree of  $\text{div } \psi$  is  $k/4$ . Thus, by Lemmas 7 and 22, for  $f = \sum_m c_m q^m \in M^1(52) \cap \mathbb{Z}((q))$ , the Borchers form  $\psi_{F_f}$  has even weight  $k$  and a trivial character if and only if

$$(9) \quad \sum_{m < 0} c_m \sum_{r \in \mathbb{Z}^+, 4m/r^2 \text{ is a discriminant}} \frac{1}{e_{4m/r^2}} |\text{CM}(4m/r^2)| = k/4$$

and

$$(10) \quad \sum_{m=-2n^2} c_m \equiv \sum_{m=-13n^2} c_m \equiv \sum_{m=-26n^2} c_m \equiv k/2 \pmod{2}.$$

Now from Example 19, we know that for each  $j \geq 3$ , there exists a unique element  $f_j$  in  $M^1(52) \cap \mathbb{Z}((q))$  such that its Fourier expansion is of the form  $f_j = q^{-j} + c_{-2}q^{-2} + c_{-1}q^{-1} + \dots$ . In particular, we find

$$\begin{aligned} f_7 &= q^{-7} - q^{-2} + 2q + \dots, \\ f_{13} &= q^{-13} - q^{-2} - 2q^{-1} + q + \dots, \\ f_{26} &= q^{-26} + q^{-1} - q + \dots. \end{aligned}$$

The modular form

$$f = f_{26} - f_{13} + 2f_7 = q^{-26} - q^{-13} + 2q^{-7} - q^{-2} + 3q^{-1} + 2q + \dots,$$

satisfies the conditions in (9) and (10) with  $k = 2$ . (Note that there do not exist CM-points of discriminants  $-4$  and  $-7$  on the Shimura curve  $X_0^{26}(1)$ , so the presence of the terms  $q^{-7}$  and  $q^{-1}$  will not contribute anything to the divisor of the Borchers form.) This proves Claim (i).

To prove Claim (ii), it suffices to show that for each discriminant  $d < 0$ , there exists a modular form  $f$  in  $M^1(52) \cap \mathbb{Z}((q))$  satisfying (9) and (10) with  $k = 0$  such that  $\text{div } \psi_{F_f} = P_d^{\times h_d} - h_d P_{-8}$ . For the special cases  $d = -52$  and  $d = -104$ , we may choose  $f$  to be  $2f_{13}$  and  $2f_{26} + 6f_7$ , respectively. If  $d \neq -52, -104$  and  $d$  is a fundamental discriminant, we choose  $f$  to be  $f_{|d|} + af_7$  with a proper integer  $a$  such that the coefficient of  $q^{-2}$  is  $-2h_d$ . (If  $4|d$ , we may choose  $f_{|d|/4} + bf_7$  instead.) Now assume that  $d$  is not a fundamental discriminant, say,  $d = d_0 n^2$  for some fundamental discriminant  $d_0$ . We let  $a$  be the integer such that the coefficient of  $q^{-2}$  in  $f = \sum_{r|n} \mu(r) f_{|d|/r^2} + af_7$  is  $-2h_d$ , where  $\mu(r)$  is the Möbius function. Then  $\text{div } \psi_{F_f} = P_d^{\times h_d} - h_d P_{-8}$ . (The case  $d_0 = -8$  needs a special treatment, but it is completely analogous.) This proves Claim (ii) and hence the proposition for the case  $(D, N) = (26, 1)$ .  $\square$

**3.3. Case of odd  $D$ .** The construction of Borchers forms in the case of odd  $D$  is a little more complicated than the case of even  $D$ . The idea of using  $\{\infty\}$ -weakly holomorphic modular forms to construct Borchers forms is no longer sufficient for our purpose. The reason is that if the divisor of a Borchers form arising from a  $\{\infty\}$ -weakly holomorphic modular form is supported at a CM-point of discriminant  $d$ ,  $d \equiv 1 \pmod{4}$ , then it also is supported at CM-points of discriminant  $4d$ . However, in practice, we are often required to construct Borchers forms whose divisors are supported at CM-points of discriminant  $d$ , but not at CM-points of discriminant  $4d$ . Thus, in the case of odd  $D$ , we will need to use  $\{\infty, 0\}$ -weakly holomorphic modular forms to construct desired Borchers forms.

Assume that  $D$  is odd and  $N$  is squarefree. As usual, we let  $\mathcal{O}$  be an Eichler order of level  $N$  in the quaternion algebra  $B$  of discriminant  $D$ , and  $L$  be the lattice formed by elements of trace 0 in  $\mathcal{O}$ . For convenience, for a modular form  $f$ , we let  $P(f)$  denote the principal part of  $f$  at  $\infty$ , i.e., the sums of the terms with negative exponents in the Fourier expansion of  $f$ . Similarly, for a vector-valued modular form  $F = \sum_{\eta \in L^\vee/L} F_\eta e_\eta$ , we let

$$P(F) = \sum_{\eta} P(F_\eta) e_\eta.$$

**Lemma 24.** *Let  $M$  be the level of  $L$ . Suppose  $f$  is a  $\{\infty, 0\}$ -weakly holomorphic scalar-valued modular of weight  $1/2$  on  $\Gamma_0(M)$  and  $F_f$  was given in Lemma 10. Assume that  $P(f|_{1/2}S) = \sum_{n>0} b_n q^{-n/M}$ . Then*

$$P(F_f) = P(f)e_0 + \frac{Me^{2\pi i/8}}{\sqrt{|L^\vee/L|}} \sum_{n>0} b_n q^{-n/M} \sum_{\eta \in L^\vee/L, \text{nm}(\eta) \in n/M + \mathbb{Z}} e_\eta.$$

*Proof.* Since  $f$  is of  $\{\infty, 0\}$ -weakly holomorphic, if  $\gamma$  is an element of  $\text{SL}(2, \mathbb{Z})$  such that  $\gamma\infty$  is not equivalent to the cusp  $\infty$  or 0, then we have  $P(f|_{1/2}\gamma) = 0$ . Now  $\gamma = I$  is the only right coset representative of  $\Gamma_0(M)$  in  $\text{SL}(2, \mathbb{Z})$  with  $\gamma\infty \sim \infty$  and  $\gamma = ST^j$ ,  $j = 0, \dots, M-1$ , are the only right coset representatives with  $\gamma\infty \sim 0$ . Thus,

$$P(F_f) = P(f)e_0 + \sum_{j=0}^{M-1} P(f|_{1/2}ST^j)\rho_L(T^{-j}S^{-1})e_0.$$

Since

$$\rho_L(S^{-1})e_\delta = \frac{e^{2\pi i/8}}{\sqrt{|L^\vee/L|}} \sum_{\eta \in L^\vee/L} e^{-\langle \eta, \delta \rangle} e_\eta,$$

we find

$$\begin{aligned} P(F_f) &= P(f)e_0 + \frac{e^{2\pi i/8}}{\sqrt{|L^\vee/L|}} \sum_{j=0}^{M-1} P(f|_{1/2}ST^j)\rho_L(T^{-j}) \sum_{\eta \in L^\vee/L} e_\eta \\ &= P(f)e_0 + \frac{e^{2\pi i/8}}{\sqrt{|L^\vee/L|}} \sum_{n>0} b_n q^{-n/M} \sum_{j=0}^{M-1} \sum_{\eta \in L^\vee/L} e^{2\pi i j(-n/M + \text{nm}(\eta))} e_\eta \\ &= P(f)e_0 + \frac{Me^{2\pi i/8}}{\sqrt{|L^\vee/L|}} \sum_{n>0} b_n q^{-n/M} \sum_{\eta \in L^\vee/L, \text{nm}(\eta) \in n/M + \mathbb{Z}} e_\eta. \end{aligned}$$

This proves the lemma.  $\square$

In general, the principal part  $e^{2\pi i/8}|L^\vee/L|^{-1/2}P(f|_{1/2}S)$  in the lemma lie in  $\mathbb{C}[q^{-1/M}]$ . For our purpose, we will only consider those  $f$  such that

$$P(f) \in \mathbb{Z}[q^{-1}], \quad \frac{Me^{2\pi i/8}}{\sqrt{|L^\vee/L|}}P(f|_{1/2}S) \in \mathbb{Z}[q^{-1/4}].$$

**Lemma 25.** *Let  $f$  be as in the lemma above. Suppose that  $P(f)$  and  $P(f|_{1/2}S)$  are of the form*

$$P(f) = \sum_{n>0, n \in \mathbb{Z}} a_n q^{-n}, \quad \frac{Me^{2\pi i/8}}{\sqrt{|L^\vee/L|}}P(f|_{1/2}S) = \sum_{n>0, n \in \mathbb{Z}} b_n q^{-n/4}$$

for some integers  $a_n$  and  $b_n$ . Then

$$\begin{aligned} \operatorname{div} \psi_{F_f} = & \sum_n a_n \sum_{r \in \mathbb{Z}^+, -4n/r^2 \text{ is a discriminant}} \frac{1}{e_{-4n/r^2}} P_{-4n/r^2} \\ & + \sum_n b_n \sum_{r \in \mathbb{Z}^+, -N^2 n/r^2 \text{ is a discriminant}} \frac{1}{e_{-N^2 n/r^2}} P_{-N^2 n/r^2}, \end{aligned}$$

where  $e_d$  is the cardinality of the stabilizer subgroup of a CM-point of discriminant  $d$  in  $N_B^+(\mathcal{O})/\mathbb{Q}^*$ .

*Proof.* Let  $q$  be a prime satisfying the condition in Lemma 8 so that  $B = \left(\frac{DN, q}{\mathbb{Q}}\right)$  is a quaternion algebra of discriminant  $D$ . Let  $\mathcal{O}$  be the Eichler order of level  $N$  spanned by  $e_1, \dots, e_4$  given in (3) and  $\{\ell_1, \ell_2, \ell_3\}$  be given as in (4). The contribution from  $P(f)e_0$  to the divisor of  $\psi_{F_f}$  is described in Lemma 22. Here we are mainly concerned with the contribution from  $P(f|S)$ .

Consider the case of odd  $N$  first. Let  $\lambda$  be an element in  $L^\vee = \mathbb{Z}\ell_1/2 + \mathbb{Z}\ell_2/DN + \mathbb{Z}\ell_3/DN$  satisfying  $\operatorname{nm}(\lambda) = n/4$  for some positive integer  $n$ . We need to determine the discriminant of the optimal embedding  $\phi : \mathbb{Q}(\sqrt{-n}) \hookrightarrow B$  that maps  $\sqrt{-n}$  to  $2\lambda$ .

Observe that  $2DN\lambda \in \mathcal{O}$  and  $\operatorname{nm}(2DN\lambda) = -D^2N^2n$ . By [1, Proposition 1.53], we must have  $2N\lambda \in \mathcal{O}$ , i.e.,  $\lambda = c_1\ell_1/2 + c_2\ell_2/N + c_3\ell_3/N$  for some integers  $c_1, c_2$ , and  $c_3$ , and the discriminant of the optimal embedding  $\phi$  is  $-4N^2n/r^2$  for some integer  $r$ .

From the Gram matrix in (5), we have

$$\operatorname{nm}(N\lambda) = -\frac{qN^2c_1^2}{4} + \frac{q-1}{4}DNc_2^2 + \frac{1-a^2DN}{q}DNc_3^2 - aDN^2c_1c_3 + DNc_2c_3.$$

As  $q$  is congruent to 1 modulo 4, this shows that  $\operatorname{nm}(2N\lambda) \equiv 0, 3 \pmod{4}$ . Therefore, if  $n \equiv 1, 2 \pmod{4}$ , then there does not exist  $\lambda \in L^\vee$  such that  $\operatorname{nm}(\lambda) = n/4$ . Also, if  $n \equiv 3 \pmod{4}$ , then  $c_1$  must be odd and

$$\frac{1+N\lambda}{2} = \frac{1-Nc_1}{2}e_1 + Nc_1e_2 + c_2e_3 + c_3e_4 \in \mathcal{O}.$$

In this case, the discriminant of the optimal embedding is  $-N^2n/r^2$  for some  $r$ . If  $n \equiv 0 \pmod{4}$ , then  $c_1$  is even. It follows that  $N\lambda \in \mathcal{O}$  and the optimal embedding has discriminant  $-N^2n/r^2$  for some  $r$ .

Conversely, given a CM-point  $\tau$  of discriminant  $-N^2n/r^2$ , there exists an element  $\lambda = d_1\ell_1 + d_2\ell_2 + d_3\ell_3 \in L$  fixing  $\tau$  and having norm

$$\operatorname{nm}(\lambda) = \begin{cases} -N^2n/4, & \text{if } n \equiv 0 \pmod{4}, \\ -N^2n, & \text{if } n \equiv 3 \pmod{4}. \end{cases}$$

Note that if  $n$  is odd, then we must have  $(1 + \lambda)/2 \in \mathcal{O}$ . In other words,  $d_2$  and  $d_3$  are even and  $d_1$  is odd. On the other hand,

$$\mathrm{nm}(\lambda) = -qd_1^2 + \frac{q-1}{4}DNd_2^2 + \frac{1-a^2DN}{q}DNd_3^2 - 2aDNd_1d_3 + DNd_2d_3.$$

Since  $N$  is squarefree, this implies that  $N|d_1$ . Setting

$$\lambda' = \begin{cases} \lambda/N, & \text{if } n \equiv 0 \pmod{4}, \\ \lambda/(2N), & \text{if } n \equiv 3 \pmod{4}, \end{cases}$$

we find  $\lambda' \in L^\vee$  with  $\mathrm{nm}(\lambda') = n/4$ . This proves the lemma for the case of odd  $N$ . The proof of the case of even  $N$  is similar and is omitted.  $\square$

**Lemma 26.** *Let  $M$  be the level of the lattice  $L$  and let  $f(\tau) = \prod_{d|M} \eta(d\tau)^{r_d}$  be an admissible eta-product. (See Definition 13.) Then we have*

$$\frac{e^{2\pi i/8}}{\sqrt{|L^\vee/L|}}(f|_{1/2}S)(\tau) = \frac{1}{\sqrt{|L^\vee/L|}} \prod_{d|M} \frac{1}{d^{r_d/2}} \eta(\tau/d)^{r_d} \in \mathbb{Q}((q^{1/M})).$$

*Proof.* The lemma follows immediately from the formula  $\eta(-1/\tau) = e^{-2\pi i/8} \sqrt{\tau} \eta(\tau)$  and the assumptions that  $\sum r_d = 1$  and that  $|L^\vee/L| \prod_{d|M} d^{r_d}$  is a square in  $\mathbb{Q}^*$ .  $\square$

**Lemma 27.** *Let  $D_0$  be the odd part of  $DN$  and  $g$  be the genus of the modular curve  $X_0(4D_0)$ .*

(1) *For nonnegative integers  $m$  and  $n$  with*

$$m + n \geq 2g - 2 - \sum_{d|D_0} \lfloor d/4 \rfloor,$$

*we have*

$$\dim_{\mathbb{C}} M_{m,n}^{!,!}(4D_0) = m + n + \sum_{d|D_0} \lfloor d/4 \rfloor + 1 - g.$$

(2) *Let  $m$  be a nonnegative integers such that  $m \geq 2g - 2 - \sum_{d|D_0} \lfloor d/4 \rfloor$ . Then for each positive integer  $n$ , there exists a modular form  $f_n$  in  $M_{m,n}^{!,!}(4D_0)$  with a pole of order  $n$  at 0. Furthermore, the space  $M^{!,!}(4D_0)$  is spanned by  $M^!(4D_0)$  and  $f_1, f_2, \dots$*

*Proof.* The proof of Part (1) is similar to that of Lemma 17 and is omitted. To prove Part (2), we notice that Part (1) implies that when  $m \geq 2g - 2 - \sum_{d|D_0} \lfloor d/4 \rfloor$ , the space  $M_{m,0}^{!,!}(4D_0)$  has co-dimension  $n$  in  $M_{m,n}^{!,!}(4D_0)$ . It follows that for each integer  $k$  with  $1 \leq k \leq n$ , there exists a modular form  $f_k$  in  $M_{m,n}^{!,!}(4D_0)$  with a pole of order  $k$  at 0. Now if  $f$  is a modular form in  $M^{!,!}(4D_0)$ , then for some linear combination  $\sum c_n f_n$ , we have  $f - \sum c_n f_n \in M^!(4D_0)$ . This proves Part (2).  $\square$

**Proposition 28.** *For  $(D, N)$  in Theorem 1 with odd  $D$  and squarefree  $N$ , the space  $M^{!,!}(4D_0)$  is spanned by admissible eta-products. Moreover, if  $f(\tau) \in M^{!,!}(4D_0) \cap \mathbb{Q}((q))$ , then*

$$\frac{e^{2\pi i/8}}{\sqrt{|L^\vee/L|}}(f|_{1/2}S)(\tau) \in \mathbb{Q}((q^{1/(4D_0)})).$$



*Proof.* Suppose that we can find an eta-product  $t(\tau)$  such that  $t(\tau)$  is a modular function on  $X_0(4D_0)$  with a unique pole at  $\infty$ . Let  $k$  be the order of pole of  $t(\tau)$  at  $\infty$ . Then  $t(-1/(4D_0\tau))$  is a modular function on  $X_0(4D_0)$  with a unique pole of order  $k$  at 0. Let  $g$  be the genus of  $X_0(4D_0)$  and  $m$  be an integer with  $m \geq 2g - 2 - \sum_{d|D_0} \lfloor d/4 \rfloor$ . By Lemma 27, for each positive integer  $j$ , there exist a modular form in  $M_{m,j}^{!,!}(4D_0)$  with a pole of order  $j$  at 0. It follows that

$$M_{m,n+k}^{!,!}(4D_0) = M_{m,n}^{!,!}(4D_0) + t(-1/4D_0\tau)M_{m,n}^{!,!}(4D_0)$$

Thus, to prove the proposition, it suffices to show that

- (i) there exists an eta-product  $t(\tau)$  such that  $t(\tau)$  is a modular function on  $\Gamma_0(4D_0)$  with a unique pole of at  $\infty$ ,
- (ii) admissible eta-products span  $M^!(4D_0)$ , and
- (iii) admissible eta-products span  $M_{m,k}^{!,!}(4D_0)$ , where  $k$  is the order of pole of  $t(\tau)$  at  $\infty$ .

For (i) and (ii), the integer programming problem involved in the construction of  $t(\tau)$  and admissible eta-products is the same as that in Proposition 18. For (iii), the integer programming problem is slightly different. For the case  $D_0 = p$  is a prime, instead of (8), we have

$$\begin{array}{rcccccccc} r_1 & + & 2r_2 & + & 4r_4 & + & pr_p & + & 2pr_{2p} & + & 4pr_{4p} & \geq & -24m \\ r_1 & + & 2r_2 & + & r_4 & + & pr_p & + & 2pr_{2p} & + & pr_{4p} & \geq & 0 \\ 4r_1 & + & 2r_2 & + & r_4 & + & 4pr_p & + & 2pr_{2p} & + & pr_{4p} & \geq & 0 \\ pr_1 & + & 2pr_2 & + & 4pr_4 & + & r_p & + & 2r_{2p} & + & 4r_{4p} & \geq & 0 \\ pr_1 & + & 2pr_2 & + & pr_4 & + & r_p & + & 2r_{2p} & + & r_{4p} & \geq & 0 \\ 4pr_1 & + & 2pr_2 & + & pr_4 & + & 4r_p & + & 2r_{2p} & + & r_{4p} & \geq & -24k \end{array}$$

where the last inequality corresponds to the condition that the order of pole at 0 is at most  $k$ . After setting up the integer programming problems, we check case by case that admissible eta-products do expand  $M^{!,!}(4D_0)$ .

Since every modular form  $f(\tau)$  in  $M^{!,!}(4D_0) \cap \mathbb{Q}((q))$  is a  $\mathbb{Q}$ -linear combination of admissible eta-products, the assertion about rationality of Fourier coefficients of  $f|_{1/2}S$  follows from Lemma 26.  $\square$

**Example 29.** Consider the Shimura curve  $X_0^{15}(1)/W_{15,1}$ . We have  $|L^\vee/L| = 450$  and the level of the lattice  $L$  is 60. By solving the relevant integer programming problem, we find that

$$t(\tau) = \frac{\eta(2\tau)\eta(12\tau)^6\eta(20\tau)^2\eta(30\tau)^3}{\eta(4\tau)^2\eta(6\tau)^3\eta(10\tau)\eta(60\tau)^6} = q^{-8} - q^{-6} + q^{-4} + q^{-2} + q^4 + \dots$$

is a modular function on  $\Gamma_0(60)$  with a unique pole of order 8 at  $\infty$ . Also, the genus of  $X_0(60)$  is 7. By Lemma 17, the number of gaps of  $M^!(60)$  is 3, and for  $n \geq 8$ , we have  $\dim M_n^!(60) = n - 2$ . According to the proof of Proposition 18, we should find an integer  $n_0$  such that  $M_{n_0+8}^!(60)$  is spanned by eta-products and for each integer  $j$  with  $n_0 < j \leq n_0 + 8$ , there exists a modular form in  $M_{n_0+8}^!(60)$  with a pole of order  $j$  at  $\infty$ . It turns out that we can choose  $n_0 = 3$ . (In other words, we will see that the gaps are 1, 2, 3.)

For convenience, we let  $(r_1, r_2, r_3, r_4, r_5, r_6, r_{10}, r_{12}, r_{15}, r_{20}, r_{30}, r_{60})$  represents the eta-product  $\prod_{d|60} \eta(d\tau)^{r_d}$ . By solving the integer programming program, we find that

there are at least 96 eta-products in  $M_{11}^1(60)$ . Among them, we choose

$$\begin{aligned} f_{11} &= (0, 1, 0, -1, -1, -1, 0, 2, 5, 2, 1, -7), & f_{10} &= (0, 0, -1, 0, 2, 1, 0, 1, 1, 1, 2, -6), \\ f_9 &= (0, 0, -1, 0, -1, 2, -1, 0, 2, 3, 4, -7), & f_8 &= (0, 0, -1, 1, -2, 0, 1, 1, 5, 1, 0, -5), \\ f_7 &= (0, 1, 1, -1, 2, -1, -2, 1, -1, 3, 3, -5), & f_6 &= (0, 1, 0, -1, 0, -1, 0, 2, 2, 1, 1, -4), \\ f_5 &= (0, 0, -1, 0, -1, 1, 2, 1, 2, 0, 0, -3), & f_4 &= (0, 0, -1, 0, 0, 2, -1, 0, -1, 2, 4, -4), \\ f_0 &= (-2, 5, 0, -2, 0, 0, 0, 0, 0, 0, 0, 0). \end{aligned}$$

They form a basis for  $M_{11}^1(60)$ . (The subscripts are the orders of poles at  $\infty$ .) Then multiplying those modular forms by suitable powers of  $t(\tau)$ , we get, for each a nongap integer  $j > 0$ , a modular form in  $M^1(60) \cap \mathbb{Z}((q))$  with a unique pole of order  $j$  at  $\infty$  and a leading coefficient 1.

Furthermore, we find that there are at least 102 eta-products in  $M_{3,8}^{!,!}(60)$ . Among them, we choose

$$\begin{aligned} g_1(\tau) &= \frac{\eta(2\tau)\eta(3\tau)\eta(4\tau)\eta(5\tau)\eta(12\tau)\eta(30\tau)}{\eta(\tau)^2\eta(6\tau)\eta(60\tau)^2} = q^{-3} + 2q^{-2} + 4q^{-1} + \cdots, \\ g_2(\tau) &= \frac{\eta(2\tau)^4\eta(3\tau)^2\eta(10\tau)^3\eta(12\tau)}{\eta(\tau)^3\eta(4\tau)\eta(5\tau)\eta(6\tau)^2\eta(20\tau)\eta(60\tau)} = q^{-2} + 3q^{-1} + 5 + 8q + \cdots, \\ g_3(\tau) &= \frac{\eta(4\tau)^2\eta(6\tau)\eta(10\tau)^2}{\eta(\tau)^2\eta(20\tau)\eta(60\tau)} = q^{-2} + 2q^{-1} + 5 + 10q + 18q^2 + \cdots, \\ g_4(\tau) &= \frac{\eta(2\tau)^3\eta(3\tau)^4\eta(5\tau)\eta(12\tau)^2\eta(30\tau)}{\eta(\tau)^4\eta(4\tau)\eta(6\tau)^3\eta(15\tau)\eta(60\tau)} = q^{-1} + 4 + 11q + 24q^2 + \cdots, \\ g_5(\tau) &= \frac{\eta(2\tau)^5\eta(3\tau)\eta(6\tau)\eta(10\tau)}{\eta(\tau)^5\eta(12\tau)\eta(60\tau)} = q^{-2} + 5q^{-1} + 15 + 39q + 90q^2 + \cdots, \\ g_6(\tau) &= \frac{\eta(2\tau)^3\eta(3\tau)^2\eta(5\tau)\eta(6\tau)^2}{\eta(\tau)^5\eta(12\tau)\eta(60\tau)} = q^{-2} + 5q^{-1} + 17 + 48q + \cdots, \\ g_7(\tau) &= \frac{\eta(2\tau)^2\eta(3\tau)\eta(4\tau)\eta(5\tau)^3\eta(6\tau)}{\eta(\tau)^5\eta(12\tau)\eta(15\tau)} = 1 + 5q + 18q^2 + 54q^3 + \cdots, \\ g_8(\tau) &= \frac{\eta(2\tau)^4\eta(3\tau)^2\eta(5\tau)^3\eta(12\tau)^2\eta(15\tau)}{\eta(\tau)^6\eta(4\tau)\eta(6\tau)^2\eta(10\tau)\eta(60\tau)} = q^{-1} + 6 + 23q + 72q^2 + \cdots \end{aligned}$$

of weight  $1/2$  on  $\Gamma_0(60)$ . By Lemma 26,

$$\begin{aligned} \frac{60e^{2\pi i/8}}{15\sqrt{2}}(g_1|S)(\tau) &= \frac{2}{3}(q^{-2/60} + 2q^{-1/60} + 4 + 8q^{1/60} + 14q^{2/60} + \cdots), \\ \frac{60e^{2\pi i/8}}{15\sqrt{2}}(g_2|S)(\tau) &= q^{-2/60} + q^{-1/60} + 2 + 4q^{1/60} + 6q^{2/60} + 8q^{3/60} + \cdots, \\ \frac{60e^{2\pi i/8}}{15\sqrt{2}}(g_3|S)(\tau) &= 2(q^{-3/60} + q^{-2/60} + 2q^{-1/60} + 4 + 6q^{1/60} + \cdots), \\ \frac{60e^{2\pi i/8}}{15\sqrt{2}}(g_4|S)(\tau) &= 2(q^{-4/60} + q^{-3/60} + q^{-2/60} + 2q^{-1/60} + 3 + \cdots), \\ \frac{60e^{2\pi i/8}}{15\sqrt{2}}(g_5|S)(\tau) &= q^{-5/60} + q^{-4/60} + 2q^{-3/60} + 3q^{-2/60} + 5q^{-1/60} + \cdots, \\ \frac{60e^{2\pi i/8}}{15\sqrt{2}}(g_6|S)(\tau) &= \frac{2}{3}(q^{-6/60} + q^{-5/60} + 2q^{-4/60} + 3q^{-3/60} + \cdots), \end{aligned}$$

$$\begin{aligned}\frac{60e^{2\pi i/8}}{15\sqrt{2}}(g_7|S)(\tau) &= \frac{1}{5}(q^{-7/60} + q^{-3/60} + q^{-2/60} + 2q^{1/60} + \dots), \\ \frac{60e^{2\pi i/8}}{15\sqrt{2}}(g_8|S)(\tau) &= \frac{2}{15}(q^{-8/60} + q^{-7/60} + 2q^{-6/60} + 3q^{-5/60} + \dots).\end{aligned}$$

Thus, letting

$$h_1 = 3g_1/2 - g_2, \quad h_2 = g_2, \quad h_3 = g_3/2, \quad h_4 = g_4/2,$$

and

$$h_5 = g_5, \quad h_6 = 3g_6/2, \quad h_7 = 5g_7, \quad h_8 = 15g_8/2,$$

we get a sequence  $h_j, j = 1, \dots, 8$ , of modular forms such that

$$\frac{60e^{2\pi i/8}}{15\sqrt{2}}(h_j|S)(\tau) = q^{-j/60} + \dots.$$

Now we have

$$t(-1/60\tau) = 5 \frac{\eta(2\tau)^3 \eta(3\tau)^2 \eta(5\tau)^6 \eta(30\tau)}{\eta(\tau)^6 \eta(6\tau) \eta(10\tau)^3 \eta(15\tau)^2} = 5 + 30q + 120q^2 + 390q^3 + \dots,$$

which is a modular function on  $\Gamma_0(60)$  having a unique pole of order 8 at the cusp 0. Thus, by multiplying  $h_j$  with suitable powers of  $t(-1/60\tau)$ , we get, for each positive integer  $m$ , an  $\{\infty, 0\}$ -weakly holomorphic modular form  $h_m$  whose order of pole at  $\infty$  is bounded by 3, while

$$\frac{60e^{2\pi i/8}}{15\sqrt{2}}(h_m|S)(\tau) = q^{-m/60} + \dots.$$

**Remark 30.** We expect that, as in the case of even  $D$ , for  $(D, N)$  in Theorem 1 with odd  $D$  and squarefree  $N$ , all meromorphic modular forms of even weights on  $X_0^D(N)/W_{D,N}$  with a divisor supported on CM-divisors can be realized as a Borcherds form. However, a proof along the line of that of Proposition 23 will be a little complicated because the Fourier expansions at 0 of a modular form in  $M^{!,!}(4D_0) \cap \mathbb{Z}((q))$  may not be integral.

**Example 31.** Here we give an example showing how to construct a Borcherds form with a desired divisor on  $X_0^{15}(1)/W_{15,1}$  using modular forms in  $M^{!,!}(60)$ .

Suppose that we wish to construct a Borcherds form with a divisor  $P_{-12} - P_{-3}$ . For a positive integer  $j$ , we let  $h_j$  be the modular form in  $M^{!,!}(60)$  constructed in Example 29 with the properties that its order of pole at  $\infty$  is bounded by 3 and

$$\frac{60e^{2\pi i/8}}{15\sqrt{2}}(h_j|S)(\tau) = q^{-j/60} + \dots.$$

A suitable linear combination of these  $h_m$  will yield a function  $h$  with

$$(11) \quad h(\tau) = q^{-2} + 11 + \dots, \quad \frac{60e^{2\pi i/8}}{15\sqrt{2}}(h|S)(\tau) = 2q^{-3/4} + 4q^{1/60} + 4q^{2/60} + \dots.$$

By Lemma 25,

$$\operatorname{div} \psi_{F_h} = \frac{1}{3}P_{-3}.$$

Let

$$f = f_8 - f_5 + f_4 = q^{-8} + 2q^{-3} + q^{-2} + 2q^2 + \dots$$

where  $f_j$  are as given in Example 29. By Lemma 25,

$$\operatorname{div} \psi_{F_f} = P_{-12} + \frac{1}{3}P_{-3}.$$

Therefore, we find that  $\psi_{F_{f-4h}}$  is a Borcherds form with a divisor  $P_{-12} - P_{-3}$ .

#### 4. EQUATIONS OF HYPERELLIPTIC SHIMURA CURVES

Recall that a compact Riemann surface  $X$  of genus  $\geq 2$  is hyperelliptic if and only if there exists a double covering  $\pi : X \rightarrow \mathbb{P}(\mathbb{C})$ , or equivalently, if there exists an involution  $w : X \rightarrow X$  such that  $X/w$  has genus zero. The involution  $w$  is unique and is called the hyperelliptic involution.

**Theorem C** ([26, Theorems 7 and 8]). *Let  $g(D, N)$  denote the genus of  $X_0^D(N)$ . The following table gives the full list of hyperelliptic Shimura curves,  $D > 1$ , and their hyperelliptic involutions.*

TABLE 1. List of hyperelliptic Shimura curves and their hyperelliptic involutions

$D$	$N$	$g(D, N)$	$w$
26	1	2	$w_{26}$
35	1	3	$w_{35}$
38	1	2	$w_{38}$
39	1	3	$w_{39}$
51	1	3	$w_{51}$
55	1	3	$w_{55}$
57	1	3	$w_{19}$
58	1	2	$w_{29}$
62	1	3	$w_{62}$
69	1	3	$w_{69}$
74	1	4	$w_{74}$
82	1	3	$w_{41}$
86	1	4	$w_{86}$
87	1	5	$w_{87}$
93	1	5	$w_{31}$
94	1	3	$w_{94}$
95	1	7	$w_{95}$
111	1	7	$w_{111}$
119	1	9	$w_{119}$
134	1	6	$w_{134}$
146	1	7	$w_{146}$
159	1	9	$w_{159}$
194	1	9	$w_{194}$
206	1	9	$w_{206}$

$D$	$N$	$g(D, N)$	$w$
6	11	3	$w_{66}$
6	17	3	$w_{34}$
6	19	3	$w_{114}$
6	29	5	$w_{174}$
6	31	5	$w_{186}$
6	37	5	$w_{222}$
10	11	5	$w_{110}$
10	13	3	$w_{65}$
10	19	5	$w_{38}$
10	23	9	$w_{230}$
14	3	3	$w_{14}$
14	5	3	$w_{14}$
15	2	3	$w_{15}$
15	4	5	$w_{15}$
21	2	3	$w_7$
22	3	3	$w_{66}$
22	5	5	$w_{110}$
26	3	5	$w_{26}$
39	2	7	$w_{39}$

**4.1. Method.** Let us briefly explain our method to compute equations of these hyperelliptic Shimura curves. Before doing that, we remark that in addition to Borcherds forms and Schofer's formula, arithmetic properties of CM-points are also crucial in our computation. We refer the reader to [17, Section 5] for an explicit description of the Shimura reciprocity law.

Let  $X_0^D(N)$  be one of the curves in Ogg's list. Since the hyperelliptic involution of  $X_0^D(N)$  is an Atkin-Lehner involution, the genus of  $X_0^D(N)/W_{D,N}$  is necessarily 0. Moreover, it turns out that any of these  $X_0^D(N)/W_{D,N}$  has at least three rational CM-points  $\tau_1$ ,  $\tau_2$ , and  $\tau_3$  of discriminants  $d_1$ ,  $d_2$ , and  $d_3$ , respectively. Thus, there is a Hauptmodul  $s(\tau)$  on  $X_0^D(N)/W_{D,N}$  with  $s(\tau_1) = \infty$ ,  $s(\tau_2) = 0$  and  $s(\tau_3) \in \mathbb{Q}$ .

Let  $W$  be a subgroup of index 2 of  $W_{D,N}$ . Suppose that  $w_m$  is an element of  $W_{D,N}$  not in  $W$ . Then  $X_0^D(N)/W \rightarrow X_0^D(N)/W_{D,N}$  is a double cover ramified at certain CM-points that are fixed points of the Atkin-Lehner involutions  $w_{mn/\gcd(m,n)^2}$ ,  $w_n \in W$ . Thus, an equation of  $X_0^D(N)/W$  is

$$(12) \quad y^2 = a \prod_{\tau \text{ ramified}, s(\tau) \neq \infty} (s - s(\tau)),$$

where  $a$  is a rational number depending on the arithmetic of  $X_0^D(N)/W$ . Specifically,  $a$  must be a rational number such that  $(a \prod_{\tau \text{ ramified}} (-s(\tau)))^{1/2}$  is in the field of definition of a CM-point of discriminant  $d_2$  on  $X_0^D(N)/W$ . As an additional check, note that when  $\tau_1$  is not a ramified point, the right-hand side of (12) is a polynomial of even degree and  $a$  must be a rational number such that  $\sqrt{a}$  is in the field of definition of a CM-point of discriminant  $d_1$  on  $X_0^D(N)/W$ .

To determine the coefficients of the polynomial on the right-hand side of (12), we simply have to know the values of  $s$  and  $y^2$  at sufficiently many points. For this purpose, we observe that  $s$  and  $y^2$  are both modular functions on  $X_0^D(N)/W_{D,N}$  with divisors supported on CM-divisors. Thus, they are both realizable as Borchers forms. (This is proved in Proposition 23 for the case of even  $D$ . We do not try to give a proof for the case of odd  $D$ , but in practice, we are always able to realize modular forms encountered as Borchers forms.) Then Schofer's formula gives us the absolute values of norms of values of  $s$  and  $y^2$  at CM-points.

In order to obtain the actual values of  $s$ , not just the absolute values, we let  $\tilde{s}$  be another Hauptmodul with  $\tilde{s}(\tau_1) = \infty$ ,  $\tilde{s}(\tau_3) = 0$ , and  $\tilde{s}(\tau_2) \in \mathbb{Q}$ . We may also realize  $\tilde{s}$  as a Borchers form. Then the absolute values of  $s(\tau_3)$  and  $\tilde{s}(\tau_2)$  obtained using Schofer's formula determine the relation  $\tilde{s} = bs + c$  between  $s$  and  $\tilde{s}$ . If  $d$  is a discriminant such that there is only one CM-point  $\tau_d$  of discriminant  $d$ , then knowing the values of  $|s(\tau_d)|$  and  $|\tilde{s}(\tau_d)| = |bs(\tau_d) + c|$  from Schofer's formula is enough to determine the value of  $s(\tau_d)$ . If there are two CM-points  $\tau_d$  and  $\tau'_d$  of discriminant  $d$ , then from the values of  $|s(\tau_d)s(\tau'_d)|$  and  $|(bs(\tau_d) + c)(bs(\tau'_d) + c)|$  we get four possible candidates for the minimal polynomial of  $s(\tau_d)$ . In almost all cases we consider, there is precisely one of the four candidates that have roots in the correct field. This gives us the values of  $s(\tau_d)$  and  $s(\tau'_d)$ . In practice, we do not need information from discriminants with more than two CM-points.

The determination of values of  $y^2$  from absolute values is easier. For example, when  $d$  is a discriminant such that there is only one CM-point of discriminant  $d$  on  $X_0^D(N)/W_{D,N}$ ,  $y(\tau_d)$  is either  $\sqrt{|y(\tau_d)^2|}$  or  $\sqrt{-|y(\tau_d)^2|}$ , but only one of them is in the correct field.

Having determined values of  $s$  and  $y^2$  at sufficiently many CM-points, it is straightforward to determine the equation of  $X_0^D(N)/W$ . Then we will either work out equations of  $X_0^D(N)/W'$  for various other subgroups  $W'$  of  $W_{D,N}$  of index 2 or use arithmetic properties of  $X_0^D(N)$  to determine equations of  $X_0^D(N)$ . We will give several examples in the next section.

## 4.2. Examples.

**Example 32.** Consider  $X_0^{15}(1)$ . In [21, Proposition 3.2.1], it is shown that an equation of  $X_0^{15}(1)$  is

$$3y^2 + (x^2 + 3)(x^2 + 243) = 0.$$

In this example, we will use Borcherds forms and Schofer's formula to obtain this result.

The curve  $X = X_0^{15}(1)$  and its various Atkin-Lehner quotients have the following geometric information.

curve	genus	elliptic points
$X$	1	$\text{CM}(-3)^{\times 2}$
$X/w_3$	0	$\text{CM}(-3)^{\times 2}, \text{CM}(-12)^{\times 2}$
$X/w_5$	1	$\text{CM}(-3)$
$X/w_{15}$	0	$\text{CM}(-3), \text{CM}(-15)^{\times 2}, \text{CM}(-60)^{\times 2}$
$X/W_{15,1}$	0	$\text{CM}(-3), \text{CM}(-12), \text{CM}(-15), \text{CM}(-60)$

According to the method described in the previous section, we should first determine the equation of  $X/W$  for some subgroup  $W$  of  $W_{15,1}$  of index 2. Here we choose  $W = \langle w_3 \rangle$ . The double cover  $X/w_3 \rightarrow X/W_{15,1}$  is ramified at the CM-points  $\tau_{-15}$  and  $\tau_{-60}$  of discriminants  $-15$  and  $-60$ . Let  $s(\tau)$  be a Hauptmodul on  $X/W_{15,1}$  taking values 0 and  $\infty$  at CM-points  $\tau_{-12}$  and  $\tau_{-3}$  of discriminants  $-12$  and  $-3$ , respectively, and satisfying  $s(\tau_{-40}) \in \mathbb{Q}$ , where  $\tau_{-40}$  is the unique CM-point of discriminant  $-40$  on  $X/W_{15,1}$ . Then an equation of  $X/w_3$  is

$$y^2 = a(s - s(\tau_{-15}))(s - s(\tau_{-60})),$$

where  $a = -3r^2$  for some  $r \in \mathbb{Q}$  since a CM-point of discriminant  $-3$  on  $X/w_3$  is defined over  $\mathbb{Q}(\sqrt{-3})$ . The divisor of  $y^2$ , as a function on  $X/W_{15,1}$ , is  $P_{-15} + P_{-60} - 2P_{-3}$ . Let also  $\tilde{s}$  be a Hauptmodul with  $\tilde{s}(\tau_{-15}) = \infty$ ,  $\tilde{s}(\tau_{40}) = 0$ , and  $\tilde{s}(\tau_{-60}) \in \mathbb{Q}$ . According to our method, we should construct Borcherds forms with divisors  $P_{-12} - P_{-3}$ ,  $P_{-40} - P_{-3}$ , and  $P_{-15} + P_{-60} - 2P_{-3}$ . A Borcherds form  $P_{-12} - P_{-3}$  is constructed in Example 31. Denote this Borcherds form by  $\psi_1$ . Here let us construct the other two Borcherds forms.

Using the notations in Example 29 and letting  $h$  be the modular form in (11), we find that

$$f_{10} - f_7 + f_5 - 2f_4 - 3h = q^{-10} - 3q^{-2} + q^{-1} - 35 + \dots$$

and

$$\frac{60e^{2\pi i/8}}{15\sqrt{2}}(f_{10} - f_7 + f_5 - 2f_4 - 3h)|S = 6q^{-3/4} + c_0 + c_1q^{1/60} + \dots$$

for some  $c_j$ . Thus, by Lemma 25, the Borcherds form  $\psi_2$  associated to this modular form has a divisor  $P_{-40} - P_{-3}$ . Also, we have

$$\begin{aligned} 2f_{15} + 4f_{13} + 2f_{12} - 2f_{10} - 4f_9 - 7f_8 - 10f_7 + 10f_6 + 3f_5 - 23f_4 - 6h \\ = 2q^{-15} - q^{-8} - 5q^{-2} - 2q^{-1} - 78 + \dots \end{aligned}$$

and

$$\begin{aligned} \frac{60e^{2\pi i/8}}{15\sqrt{2}}(2f_{15} + 4f_{13} + 2f_{12} - 2f_{10} - 4f_9 - 7f_8 - 10f_7 + 10f_6 + 3f_5 - 23f_4 - 6h)|S \\ = 12q^{-3/4} + c'_0 + c'_1q^{1/60} + \dots \end{aligned}$$

for some  $c'_j$ . Therefore, the Borchers form  $\psi_3$  associated to this modular form has a divisor  $P_{-15} + P_{-60} - 2P_{-3}$ . An application of Schofer's formula yields the following values of Borchers forms at CM-points.

	-3	-7	-12	-15	-40	-43	-60
$ \psi_1 $	$\infty$	1	0	3	1/2	1/16	1/27
$5^{-3/2} \psi_2 $	$\infty$	1/9	1/27	5/27	0	1/24	25/3 <sup>6</sup>
$ \psi_3 $	$\infty$	35/3 <sup>6</sup>	5/2 <sup>4</sup> 3 <sup>5</sup>	0	5 <sup>4</sup> /2 <sup>6</sup> 3 <sup>6</sup>	43 <sup>1</sup> 5 <sup>1</sup> 7 <sup>2</sup> /2 <sup>12</sup> 3 <sup>6</sup>	0

Observe that multiplying  $\psi_j$  by a scalar of absolute value 1 does not change the absolute value of its value at a CM-point. Thus, we may as well assume that  $\psi(\tau_{-15}) = -3$ ,  $5^{-3/2}\psi_2(\tau_{-15}) = 5/27$ , and  $\psi_3(\tau_{-7}) = -35/3^6$ . Also, we choose  $s$ ,  $\tilde{s}$ , and  $y$  such that  $s(\tau_{-15}) = -243$ ,  $\tilde{s}(\tau_{-15}) = 5$ , and  $y(\tau_{-7})^2 = -2^4 3^4 7$ . Therefore, we have

$$s = 81\psi_1, \quad \tilde{s} = 27 \cdot 5^{-3/2}\psi_2, \quad y^2 = \frac{2^4 3^{10}}{5}\psi_3.$$

Then from the table above, we obtain

$$|s(\tau_{-12})| = 0, \quad |\tilde{s}(\tau_{-12})| = 1, \quad |s(\tau_{-40})| = 81/2, \quad |\tilde{s}(\tau_{-40})| = 0,$$

which implies that  $\tilde{s}$  is equal to one of  $\pm 2s/81 \pm 1$ . As  $s(\tau_{-15}) = -243$  and  $\tilde{s}(\tau_{-15}) = 5$ , we find that  $\tilde{s} = -2s/81 - 1$ . Then the table above and the requirement that  $y(\tau_d)$  must lie in the correct field yield

	-3	-7	-12	-15	-40	-43	-60
$s$	$\infty$	81	0	-243	-81/2	81/16	-3
$\tilde{s}$	$\infty$	-3	-1	5	0	-9/8	-25/27
$y^2$	$\infty$	-2 <sup>4</sup> 3 <sup>4</sup> 7	-3 <sup>5</sup>	0	-3 <sup>4</sup> 5 <sup>3</sup> /4	-3 <sup>4</sup> 7 <sup>2</sup> 43/2 <sup>8</sup>	0

It follows that an equation of  $X/w_3$  is  $3y^2 + (s + 243)(s + 3) = 0$ .

Furthermore, the double cover  $X/w_{15} \rightarrow X/W_{15,1}$  is ramified at CM-points of discriminants -3 and -12. Thus, an equation of  $X/w_{15}$  is  $x^2 = bs$  for some  $b$ . As CM-points of discriminant -7 are rational points on  $X/w_{15}$ , we find that  $b$  must be a square, which we may assume to be 1. That is, we have  $s = x^2$ . Therefore, we have  $3y^2 + (x^2 + 243)(x^2 + 3) = 0$ , which can be taken to be an equation of  $X$ , agreeing with Jordan's result.

We remark that Elkies [11] has used Schwarzian differential equations to compute numerically the values of  $s$  at many CM-points. (His modular function differs from our  $s$  by a factor of -3.) Using Borchers forms, we verify that all the entries in Table 6 of [11] are correct.

**Example 33.** Consider the Shimura curve  $X = X_0^{26}(1)$ . In [16], González and Rotger proved that an equation of  $X$  is

$$y^2 = -2x^6 + 19x^4 - 24x^2 - 169.$$

In this example, we will obtain this result using Borchers forms.

We have the following informations about  $X$  and its Atkin-Lehner quotients.

curve	genus	elliptic points
$X$	2	none
$X/w_2$	1	$\text{CM}(-8)^{\times 2}$
$X/w_{13}$	1	$\text{CM}(-52)^{\times 2}$
$X/w_{26}$	0	$\text{CM}(-104)^{\times 6}$
$X/W_{26,1}$	0	$\text{CM}(-8), \text{CM}(-52), \text{CM}(-104)^{\times 3}$

The double cover  $X/w_{13} \rightarrow X/W_{26,1}$  is ramified at the CM-point of discriminant  $-8$  and the three CM-points of discriminant  $-104$ . Let  $s$  be a Hauptmodul on  $X/W_{26,1}$  with  $s(\tau_{-8}) = \infty$ ,  $s(\tau_{-52}) = 0$ , and  $s(\tau_{-11}) \in \mathbb{Q}$ . Then an equation of  $X/w_{13}$  is

$$y^2 = a \prod_{\tau: \text{CM-points of discriminant } -104} (s - s(\tau))$$

for some nonzero rational number  $a$ . As a modular function on  $X/W_{26,1}$ , we have  $\text{div } y^2 = P_{-104} - 3P_{-8}$ . Let  $\tilde{s}$  be another Hauptmodul on  $X/W_{26,1}$  with  $\tilde{s}(\tau_{-8}) = \infty$ ,  $\tilde{s}(\tau_{-11}) = 0$ , and  $\tilde{s}(\tau_{-52}) \in \mathbb{Q}$ . We now realize  $s$ ,  $\tilde{s}$ , and  $y^2$  as Borchers forms.

Let  $f_j$  be modular forms in  $M^!(52) \cap \mathbb{Z}((q))$  with a pole of order  $j$  at  $\infty$  and a leading coefficient 1 constructed in Example 19. Using these  $f_j$ , we find three modular forms

$$\begin{aligned} g_1 &= 2q^{-13} - 2q^{-2} - 4q^{-1} + 2q - 2q^2 - 2q^3 + \cdots, \\ g_2 &= q^{-11} + 2q^{-7} - 2q^{-2} + 4q + 4q^4 + \cdots, \\ g_3 &= 2q^{-26} + 6q^{-7} - 6q^{-2} + 2q^{-1} + 10q - 8q^2 + \cdots \end{aligned}$$

in  $M^!(52)$ . Let  $\psi_j$ ,  $j = 1, 2, 3$ , be the Borchers forms associated to  $g_j$ . By Lemma 22,

$$\text{div } \psi_1 = P_{-52} - P_{-8}, \quad \text{div } \psi_2 = P_{-11} - P_{-8}, \quad \text{div } \psi_3 = P_{-104} - 3P_{-8}.$$

Thus,  $\psi_j$  are scalar multiples of  $s$ ,  $\tilde{s}$ , and  $y^2$ , respectively. Applying Schofer's formula, we get

	-8	-11	-19	-20	-24	-52	-67
$ \psi_1 $	$\infty$	1	9	5	3	0	$81/25$
$ \psi_2 $	$\infty$	0	64	32	32	8	$2^6 7/5^2$
$13^{-3}  \psi_3 $	$\infty$	$2^{10} 11$	$2^{10} 19$	$2^{12}$	$2^{13}$	$2^6 13^5$	$2^{10} 41^2 67/5^6$

Since multiplying  $\psi_j$  by a suitable factor of absolute value 1 does not change the absolute value of its value at a CM-point, we may as well assume that  $\psi_1(\tau_{-11}) = 1$ ,  $\psi_2(\tau_{-52}) = 8$ , and  $\psi_3(\tau_{-11}) = -2^{10} 11^1 13^3$ . Also, we choose  $s$ ,  $\tilde{s}$ , and  $y$  in a way such that  $s(\tau_{-11}) = 1$ ,  $\tilde{s}(\tau_{-52}) = 1$ , and  $y(\tau_{-11})^2 = -2^4 11$ , i.e.,  $s = \psi_1$ ,  $\tilde{s} = \psi_2/8$ , and  $y^2 = \psi_3/2^6 13^3$ . Then we have  $\tilde{s} = 1 - s$  and from the table above we get

	-8	-11	-19	-20	-24	-52	-67
$s$	$\infty$	1	9	5	-3	0	$81/25$
$\tilde{s} = 1 - s$	$\infty$	0	-8	-4	4	1	$-56/25$
$y^2$	$\infty$	$-2^4 11$	$-2^4 19$	$-2^6$	$2^7$	$-13^2$	$-2^4 41^2 67/5^6$



(The signs of  $y(\tau_d)^2$  are determined by the Shimura reciprocity law.) From the data, we easily deduce that the relation between  $y$  and  $s$  is

$$y^2 = -2s^3 + 19s^2 - 24s - 169,$$

which is an equation for  $X_0^{26}(1)/w_{13}$ .

On the other hand, the cover  $X_0^{26}(1)/w_{26} \rightarrow X_0^{26}(1)/W_{26,1}$  is ramified at the CM-points of discriminants  $-8$  and  $-52$ . Thus, there is a modular function  $x$  on  $X_0^{26}(1)/w_{26}$  with  $x^2 = cs$  for some rational number  $c$ . Since CM-points of discriminant  $-11$  are rational points on  $X_0^{26}(1)/w_{26}$ , we conclude that  $c$  can be chosen to be 1. Hence  $y^2 = -2x^6 + 19x^4 - 24x^2 - 169$  is an equation for  $X_0^{26}(1)$  and the Atkin-Lehner involutions are given by

$$w_2 : (x, y) \mapsto (-x, -y), \quad w_{26} : (x, y) \mapsto (x, -y).$$

**Example 34.** Consider  $X = X_0^{111}(1)$ . We have the following informations.

curve	genus	elliptic points
$X$	7	none
$X/w_3$	4	none
$X/w_{37}$	3	$\text{CM}(-148)^{\times 4}$
$X/w_{111}$	0	$\text{CM}(-111)^{\times 8}, \text{CM}(-444)^{\times 8}$
$X/W_{111,1}$	0	$\text{CM}(-148)^{\times 2}, \text{CM}(-111)^{\times 4}, \text{CM}(-444)^{\times 4}$

Let  $s$  and  $\tilde{s}$  be modular functions on  $X/W_{111,1}$  such that  $s(\tau_{-15}) = \tilde{s}(\tau_{-15}) = \infty$ ,  $s(\tau_{-60}) = 0$ ,  $\tilde{s}(\tau_{-24}) = 0$ ,  $s(\tau_{-24}) = 1$ , and  $\tilde{s}(\tau_{-60}) = 1$ , so that  $\tilde{s} = 1 - s$ . Then an equation for  $X/w_{37}$  is

$$(13) \quad y^2 = a \prod_{\tau \in \text{CM}(-111), \text{CM}(-444)} (s - s(\tau)).$$

As CM-points of discriminant  $-60$  on  $X/w_{37}$  lie in  $\mathbb{Q}(\sqrt{-3})$ , we choose  $y$  such that  $y(\tau_{-60})^2 = -27$ . Then realizing  $s$ ,  $\tilde{s}$ , and  $y^2$  as Borcherds forms and using Schofer's formula, we deduce the following values of these modular functions at rational CM-points.

	-15	-19	-24	-43	-51	-60	-163	-267	-555
$s$	$\infty$	3	1	-3	-1	0	3/5	1/3	5
$y^2$	$\infty$	$-2^8 3^2 19$	$-2^8 3$	$-2^8 3^2 43$	$-2^8 3$	-27	$-2^8 3^2 13^2 163/5^8$	$-2^8 13^2/3^7$	$-2^8 3^1 37^2$

As the right-hand side of (13) is a polynomial of degree 8, these CM-values are not sufficient to determine the equation and we will need values of  $s$  and  $y^2$  at some degree 2 CM-points.

Let  $\tau_{-39}$  and  $\tau'_{-39}$  be the two CM-points of discriminant  $-39$  on  $X/W_{111,1}$ . Schofer's formula yields

$$|s(\tau_{-39})s(\tau'_{-39})| = 3, \quad |(1 - s(\tau_{-39}))(1 - s(\tau'_{-39}))| = 4.$$

From the Shimura reciprocity law, we know that  $s(\tau_{-39}) \in \mathbb{Q}(\sqrt{-3})$ . Thus,

$$s(\tau_{-39})s(\tau'_{-39}) = 3, \quad (1 - s(\tau_{-39}))(1 - s(\tau'_{-39})) = 4.$$

From these, we deduce that  $s(\tau_{-39}) = \pm\sqrt{-3}$ . Likewise, we find that the values of  $s$  at the two CM-points  $\tau_{-52}, \tau'_{-52}$  of discriminants  $-52$  are  $1 \pm 2\sqrt{-1}$ . Also, we have

$$y(\tau_{-39})^2 y(\tau'_{-39})^2 = 2^{16} 3^2 13, \quad y(\tau_{-52})^2 y(\tau'_{-52})^2 = 2^{16} 13^2.$$

These data are enough to determine the equation of  $X/w_{37}$ . We find that it is

$$(14) \quad y^2 = -(3s^4 - 6s^3 + 28s^2 - 10s + 1)(s^4 - 2s^3 + 4s^2 + 18s + 27).$$

Similarly, we can compute an equation for  $X/w_{111}$  by observing that  $X/w_{111} \rightarrow X/W_{111,1}$  is ramified at the two CM-points of discriminant  $-148$ , constructing a Borcherds form with divisor  $P_{-148} - 2P_{-15}$ , and evaluating at various CM-points and obtain

$$t^2 = 5s^2 - 18s + 45.$$

The conic has rational points  $(s, t) = (3, \pm 6)$  corresponding the two CM-points of discriminant  $-19$  on  $X/w_{111}$ , so it admits a rational parameterization. Specifically, let  $x$  be a Hauptmodul on  $X/w_{111}$  that has a pole and a zero at the two CM-points of discriminant  $-19$ , respectively, and takes rational values at CM-points of discriminant  $-43$ . (In terms of  $(s, t)$ , the coordinates are  $(-3, \pm 12)$ .) Then

$$x = \frac{c(s-3)}{s-t+3}$$

for some rational number  $c$ . Choose  $c = 2$  so that it takes values  $\pm 1$  at the CM-points of discriminant  $-43$ . We have

$$(s, t) = \left( \frac{3x^2 - 3x - 3}{x^2 + x - 1}, \frac{6x^2 + 6}{x^2 + x - 1} \right).$$

Plugging in  $s = (3x^2 - 3x - 3)/(x^2 + x - 1)$  in (14) and making a slight change of variables, we find that an equation of  $X_0^{111}(1)$  is

$$\begin{aligned} z^2 = & -(x^8 - 3x^5 - x^4 + 3x^3 + 1) \\ & \times (19x^8 - 44x^7 - 16x^6 + 55x^5 + 37x^4 - 55x^3 - 16x^2 + 44x + 19) \end{aligned}$$

with the actions of the Atkin-Lehner involutions given by

$$w_{37} : (x, z) \mapsto \left( -\frac{1}{x}, \frac{z}{x^8} \right), \quad w_{111} : (x, z) \mapsto (x, -z).$$

**Example 35.** Consider  $X = X_0^{146}(1)$ . Let  $s$  be the Hauptmodul of  $X/W_{146,1}$  such that  $s(\tau_{-43}) = 0$ ,  $s(\tau_{-11}) = \infty$ , and  $s(\tau_{-20}) = 1$ . Let  $y$  be a modular function on  $X/w_{73}$  such that  $y^2$  is a modular function on  $X/W_{146,1}$  with  $\text{div } y^2 = P_{-584} - 8P_{-11}$ . Realizing  $s$  and  $y^2$  as Borcherds forms and suitably scaling  $y^2$ , we find that an equation for  $X/w_{73}$  is

$$(15) \quad y^2 = -11s^8 + 82s^7 - 309s^6 + 788s^5 - 1413s^4 + 1858s^3 - 1803s^2 + 1240s - 688.$$

Similarly, we find that an equation for  $X/w_{146}$  is  $t^2 = s^2 + 4$ , where the roots of  $s^2 + 4$  correspond the to CM-points of discriminant  $-292$ . We choose a rational parameterization of the conic to be

$$(s, t) = \left( \frac{x^2 - 1}{x}, \frac{x^2 + 1}{x} \right),$$

where  $x$  is actually a modular function on  $X/w_{146}$  that has a pole and a zero at the two CM-points of discriminant  $-11$  and is equal to  $\pm 1$  at the two CM-points of discriminant  $-43$  on  $X/w_{146}$ . Substituting  $s = (x^2 - 1)/x$  in (15) and making a change of variables, we find that an equation for  $X$  is

$$\begin{aligned} z^2 = & -11x^{16} + 82x^{15} - 221x^{14} + 214x^{13} + 133x^{12} - 360x^{11} - 170x^{10} + 676x^9 \\ & - 150x^8 - 676x^7 - 170x^6 + 360x^5 + 133x^4 - 214x^3 - 221x^2 - 82x - 11, \end{aligned}$$

where the Atkin-Lehner involutions are given by

$$w_{73} : (x, y) \mapsto \left(-\frac{1}{x}, \frac{y}{x^8}\right), \quad w_{146} : (x, y) \mapsto (x, -y).$$

**Example 36.** Let  $X = X_0^{14}(5)$ . Let  $s$  be the Hauptmodul of  $X/W_{14,5}$  such that  $s(\tau_{-4}) = \infty$ ,  $s(\tau_{-11}) = 1$ , and  $s(\tau_{-35}) = 0$ . We find that an equation for  $X/\langle w_5, w_7 \rangle$  is

$$y^2 = -16s^3 - 347s^2 + 222s - 35,$$

which is isomorphic to the elliptic curve  $E_{14A5}$  in Cremona's table [10]. (In fact, we can use Cerednik-Drinfeld theory of  $p$ -adic uniformization of Shimura curves [6] to determine the singular fibers of  $X/\langle w_5, w_7 \rangle$  and conclude that it is isomorphic to  $E_{14A5}$ .) The double cover  $X/\langle w_5, w_{14} \rangle \rightarrow X/W_{14,5}$  is ramified at the CM-point of discriminant  $-4$  and the CM-point of discriminant  $-35$ , so that there is a Hauptmodul  $t$  of  $X/\langle w_5, w_{14} \rangle$  such that  $t^2 = cs$  for some rational number  $c$ . In addition, the CM-points of discriminant  $-11$  on  $X/\langle w_5, w_{14} \rangle$  are rational points. Thus, we may choose  $c = 1$  and find that an equation for  $X/w_5$  is

$$(16) \quad y^2 = -16t^6 - 347t^4 + 222t^2 - 35.$$

We next determine an equation of  $X/w_{14}$ . The double cover  $X/w_{14} \rightarrow X/\langle w_5, w_{14} \rangle$  is ramified at the two CM-points of discriminant  $-280$ . Using Schofer's formula, we find  $s(\tau_{-280}) = 5/16$  and thus, an equation for  $X/w_{14}$  is  $u^2 = d(16t^2 - 5)$  for some rational number. The point such that  $t = 0$  is the CM-point of discriminant  $-35$ . Therefore, we may choose  $d = -1$  and find that an equation for  $X/w_{14}$  is

$$u^2 + 16t^2 = 5.$$

This is a conic with rational points and a rational parameterization is

$$(t, u) = \left( \frac{x^2 - x - 1}{2x^2 + 2}, \frac{x^2 + 4x - 1}{x^2 + 1} \right).$$

Substituting  $t = (x^2 - x - 1)/(2x^2 + 2)$  into (16) and making a change of variables, we conclude that an equation for  $X_0^{14}(5)$  is

$$z^2 = -23x^8 - 180x^7 - 358x^6 - 168x^5 - 677x^4 + 168x^3 - 358x^2 + 180x - 23,$$

on which the actions of the Atkin-Lehner operators are given by

$$w_2 : (x, z) \mapsto \left(-\frac{1}{x}, \frac{z}{x^4}\right), \quad w_{14} : (x, z) \mapsto (x, -z),$$

and

$$w_{35} : (x, z) \mapsto \left( \frac{x+2}{2x-1}, \frac{25z}{(2x-1)^4} \right).$$

Note that  $X_0^{14}(5)/w_{14}$  is an example of Shimura curves of genus zero that is isomorphic to  $\mathbb{P}^1$  over  $\mathbb{Q}$  but none of the rational points is a CM-point.

**Example 37.** Let  $X = X_0^{10}(19)$ . Let  $s$  be the Hauptmodul of  $X/W_{10,19}$  such that  $s(\tau_{-8}) = 0$ ,  $s(\tau_{-40}) = \infty$ , and  $s(\tau_{-3}) = 1$ . We find that an equation for  $X/\langle w_2, w_{95} \rangle$  is

$$y^2 = -8s^3 + 57s^2 - 40s + 16,$$

which is isomorphic to the elliptic curve  $E_{190A1}$  in Cremona's table [10]. Also, the double cover  $X/\langle w_5, w_{38} \rangle \rightarrow X/W_{10,19}$  is ramified at the CM-point of discriminant  $-8$  and the CM-point of discriminant  $-40$ . The CM-points of discriminant  $-3$  are rational points on  $X/\langle w_5, w_{38} \rangle$ . Thus, arguing as before, we deduce that an equation for  $X/w_{190}$  is

$y^2 = -8x^6 + 57x^4 - 40x^2 + 16$ . Moreover, the double cover  $X/w_{38} \rightarrow X/\langle w_5, w_{38} \rangle$  is ramified at the two CM-points of discriminant  $-760$ . Since  $s(\tau_{760}) = 32/5$  and the point with  $s = 0$  is a CM-point of discriminant  $-8$ , we see that an equation for  $X/w_{38}$  is  $z^2 = 5x^2 - 32$ . We conclude that an equation for  $X$  is

$$\begin{cases} y^2 = -8x^6 + 57x^4 - 40x^2 + 16, \\ z^2 = 5x^2 - 32, \end{cases}$$

with the actions of the Atkin-Lehner involutions given by

$$\begin{aligned} w_2 : (x, y, z) &\mapsto (-x, y, z), \\ w_5 : (x, y, z) &\mapsto (x, -y, -z), \\ w_{19} : (x, y, z) &\mapsto (-x, -y, z). \end{aligned}$$

Note that as the conic  $z^2 = 5x^2 - 32$  has only real points, but no rational points, the Shimura curve  $X$  is hyperelliptic over  $\mathbb{R}$ , but not over  $\mathbb{Q}$ .

**Remark 38.** In [26], Ogg mentioned that  $X_0^{10}(19)$  and  $X_0^{14}(5)$  are the only two hyperelliptic curves that he could not determine whether they are hyperelliptic over  $\mathbb{Q}$ . Our computation shows that  $X_0^{14}(5)$  is hyperelliptic over  $\mathbb{Q}$  because the curve  $X_0^{14}(5)/w_{14}$  has rational points, but  $X_0^{10}(19)$  is not hyperelliptic over  $\mathbb{Q}$ .

**Remark 39.** Note that there is a curve, namely,  $X = X_0^{15}(4)$ , whose equation is not obtained using our method. This is because the normalizer of the Eichler order in this case is larger than the Atkin-Lehner group. For this special curve, we use the result of Tu [29]. In Lemma 13 of [29], it is shown that there is a Hauptmodul  $t_4$  on  $X/\langle w_3, w_5 \rangle$  that takes values  $\pm 1/\sqrt{-3}$ ,  $\pm\sqrt{-15}/5$  and  $(\pm 1 \pm \sqrt{-15})/8$  at CM-points of discriminants  $-12$ ,  $-15$ , and  $-60$ , respectively. Since the double cover  $X/w_3 \rightarrow X/\langle w_3, w_5 \rangle$  ramifies at CM-points of discriminants  $-15$  and  $-60$ , while the cover  $X/w_{15} \rightarrow X/\langle w_3, w_5 \rangle$  ramifies at CM-points of discriminant  $-12$ , we find that there are rational numbers  $a$  and  $b$  such that the equations of  $X/w_3$  and  $X/w_{15}$  are

$$y^2 = a(4t_4^2 - t_4 + 1)(4t_4^2 + t_4 + 1)(5t_4^2 + 3), \quad z^2 = b(3t_4^2 + 1),$$

respectively. To determine the constants  $a$  and  $b$ , we further recall that Lemma 13 of [29] shows that there is a Hauptmodul  $t_2$  on  $X_0^{15}(2)/\langle w_3, w_5 \rangle$  with

$$t_2 = \frac{5t_4^2 + 2t_4 + 1}{7t_4^2 - 2t_4 + 3}.$$

From this, the CM-values of  $t_2$  obtained using Schofer's formula, and arithmetic properties of CM-points, we see that we can choose  $a = b = -1$ . Note that  $X_0^{15}(4)$  is one of the hyperelliptic Shimura curves that are not hyperelliptic over  $\mathbb{R}$  (see [26]).

**4.3. Additional examples.** In the previous section, we determine the equations of hyperelliptic Shimura curves  $X_0^D(N)$  whose Atkin-Lehner involutions act as hyperelliptic involutions. In particular, the curves  $X_0^D(N)/W_{D,N}$  are of genus 0, so that Lemma 7 applies and we have a simple criterion for a Borcherds form to have a trivial character. Throughout this section, we make the following assumption.

**Assumption 40.** The criterion for a Borcherds form to have a trivial character is also valid for the case when  $N_B^+(\mathcal{O}) \backslash \mathfrak{H}$  has a positive genus.

**Remark 41.** Recall that a Fuchsian group of the first kind is generated by some elements  $\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g, \gamma_1, \dots, \gamma_n$  with defining relations

$$[\alpha_1, \beta_1] \dots [\alpha_g, \beta_g] \gamma_1 \dots \gamma_n = 1, \quad \gamma_i^{k_i} = 1, \quad i = 1, \dots, n,$$

where  $\alpha_j, \beta_j$  are hyperbolic elements,  $[\alpha_j, \beta_j]$  denotes the commutator,  $g$  is the genus, and  $k_i$  is an integer  $\geq 2$  or  $\infty$ . (See, for instance, [22].) Let  $\chi$  be the character of a Borchers form on  $N_B^+(\mathcal{O}) \backslash \mathfrak{H}$ . The proof of Lemma 7 given in [31] shows that  $\chi(\gamma_i) = 1$  for all  $i$  if and only if the condition in Lemma 7 holds. Thus, what we really assume in Assumption 40 is that for all hyperbolic elements  $\alpha$ , we have  $\chi(\alpha) = 1$ .

It turns out that sometimes our methods can also be used to determine equations of  $X_0^D(N)/W_{D,N}$  even when they have positive genera, under Assumption 40. However, the method becomes less systematic and it is not clear whether our methods will always work in general, so we will only give two examples in this section.

**Example 42.** Let  $X = X_0^{142}(1)/W_{142,1}$ . It is of genus 1 and has rational points (for instance, the CM-point of discriminant  $-3$ ). Thus,  $X$  is a rational elliptic curve. From the Jacquet-Langlands correspondence, we know that it must lie in the isogeny class 142A in Cremona's table [10], whose corresponding cusp form on  $\Gamma_0(142)$  has eigenvalues  $-1$  for the Atkin-Lehner involutions  $w_2$  and  $w_{71}$ . Since the isogeny class contains only one curve, we immediately conclude that the equation for  $X$  is  $E_{142A1} : y^2 + xy + y = x^3 - x^2 - 12x + 15$ . Here we will use our method to get the same conclusion. An advantage of our method is that we can determine the coordinates of all CM-points on the curve. In the next section, we will discuss the heights of these CM-points and verify Zhang's formula [32] for heights of CM-points in this particular case.

By finding many suitable eta-products, we construct 4 modular forms  $f_1, f_2, f_3, f_4$  in  $M^1(284)$  with Fourier expansions

$$\begin{aligned} f_1 &= -2q^{-87} - 2q^{-71} - 2q^{-48} - 2q^{-36} + 2q^{-16} - 2q^{-15} - 2q^{-12} + 2q^{-9} \\ &\quad - 2q^{-7} - 2q^{-3} + 2q^{-2} + 2q^{-1} - 4q \dots, \\ f_2 &= 2q^{-116} - q^{-87} - q^{-79} + 2q^{-71} + 2q^{-60} - 2q^{-48} + q^{-43} - 2q^{-29} \\ &\quad + q^{-19} - 4q^{-15} - 2q^{-12} + 2q^{-7} - 2q^{-3} - 4q + \dots, \\ f_3 &= q^{-87} - 2q^{-79} + q^{-76} - 2q^{-71} + 2q^{-48} - q^{-40} + 3q^{-32} - 2q^{-20} \\ &\quad - q^{-19} - 2q^{-12} + 2q^{-10} + q^{-8} - 2q^{-7} - 2q^{-2} + 4q + \dots, \\ f_4 &= -q^{-79} + q^{-76} + q^{-48} - q^{-40} + q^{-32} - q^{-20} - q^{-12} + q^{-10} + q^{-6} \\ &\quad - q^{-5} - q^{-2} - q^7 + \dots. \end{aligned}$$

Let  $\psi_j, j = 1, \dots, 4$ , be the Borchers form associated to  $f_j$ . Under Assumption 40, these Borchers forms have trivial characters. We have

$$\begin{aligned} \operatorname{div} \psi_1 &= P_{-4} + P_{-8} - 2P_{-3}, & \operatorname{div} \psi_2 &= P_{-19} + P_{-43} - 2P_{-3}, \\ \operatorname{div} \psi_3 &= P_{-8} + P_{-40} - 2P_{-20}, & \operatorname{div} \psi_4 &= P_{-19} + P_{-24} - 2P_{-20}. \end{aligned}$$

It is easy to show that  $\psi_2$  is a polynomial of degree 1 in  $\psi_1$  and  $\psi_4$  is a polynomial of degree 1 in  $\psi_3$ . Thus, there are modular functions  $x$  and  $y$  on  $X$  such that  $x$  has a double pole at  $\tau_{-3}$  with  $x(\tau_{-4}) = x(\tau_{-8}) = 0$  and  $x(\tau_{-19}) = x(\tau_{-43}) = 1$  and  $y$  has a double pole at  $\tau_{-20}$  with  $y(\tau_{-8}) = y(\tau_{-20}) = 0$  and  $y(\tau_{-19}) = y(\tau_{-24}) = 1$ . Computing singular moduli using Schofer's formula and choosing proper scalars of modulus 1 for  $\psi_j$ , we find

$$x = 2^{-10}\psi_1, \quad 1 - x = \psi_2, \quad y = \psi_3/2, \quad 1 - y = \psi_4/2,$$

and the values of  $x$  and  $y$  at various CM-points are

	-3	-4	-8	-19	-20	-24	-40	-43	-148	-232
$x$	$\infty$	0	0	1	-1	1/2	-1/2	1	-1	-1/2
$y$	2	1/2	0	1	$\infty$	1	0	3/2	-2	-5

Since  $y(\tau_{-4}) \neq y(\tau_{-8})$ ,  $y$  cannot lie in  $\mathbb{C}(x)$ . Therefore,  $x$  and  $y$  generate the field of modular functions on  $X$ . From the table above, we determine that the relation between  $x$  and  $y$  is

$$2(x+1)^2y^2 - (8x^2 + 11x + 1)y + 4x(2x+1) = 0.$$

Set

$$x_1 = -\frac{2(x+1)^2y - 5x^2 - 3x - 1}{x^2},$$

$$y_1 = -\frac{(4x^3 + 6x^2 - 2)y - 5x^3 - 6x^2 + x + 1}{x^3}.$$

We find  $y_1^2 + x_1y_1 + y_1 = x_1^3 - x_1^2 - 12x_1 + 15$ , which is indeed the elliptic curve  $E_{142A1}$ . The coordinates of the CM-points above on this model are

-3	-4	-8	-19	-20	-24	-40	-43	-148	-232
$-Q$	$-2Q$	$O$	$Q$	$2Q$	$3Q$	$4Q$	$-3Q$	$-4Q$	$-6Q$

where  $Q = (1, 1)$  generates the group of rational points on  $E_{142A1}$ .

**Example 43.** We next consider  $X = X_0^{302}(1)/W_{302,1}$ , which has genus 2. We can construct four modular forms  $f_1, \dots, f_4$  in  $M^1(604)$  whose associated Borcherds forms  $\psi_1, \dots, \psi_4$  have divisors

$$\operatorname{div} \psi_1 = P_{-43} + P_{-72} - P_{-19} - P_{-88},$$

$$\operatorname{div} \psi_2 = P_{-20} + P_{-36} - P_{-19} - P_{-88},$$

$$\operatorname{div} \psi_3 = P_{-8} + 2P_{-40} - P_{-4} - 2P_{-88},$$

$$\operatorname{div} \psi_4 = P_{-11} + P_{-19} + P_{-43} - P_{-4} - 2P_{-88},$$

respectively. In addition, under Assumption 40, they have trivial characters. Thus,  $\psi_1$  generates the unique genus-zero subfield of degree 2 of the hyperelliptic function field, and  $\psi_2$  is a polynomial of degree 1 in  $\psi_1$ . Also,  $\psi_4$  must be a polynomial of degree 1 in  $\psi_3$ . To see this, we observe that there exists a suitable linear combination  $a\psi_3 + b\psi_4$  such that it is a function of degree less than or equal to 2 on  $X$  and hence is contained in  $\mathbb{C}(\psi_1)$ . If this linear combination is not a constant function, then it must have a pole at  $\tau_{-88}$ ; otherwise it will have only a pole of order 1 at  $\tau_{-4}$ , which is impossible. It follows that  $\tau_{-19}$  is also a pole of this linear combination. However,  $\tau_{-19}$  can never be a pole of this function. Therefore, we conclude that this linear combination is a constant function.

Let  $x$  be the unique function on  $X$  with  $\operatorname{div} x = \operatorname{div} \psi_1$  and  $x(\tau_{-20}) = 2$  and  $y$  be the unique function with  $\operatorname{div} y = \operatorname{div} \psi_3$  and  $y(\tau_{-11}) = 1$ . Computing using Schofer's formula, we find

$d$	-4	-8	-11	-19	-20	-40	-43	-88	-148	-232
$x$	-1	3/2	1	$\infty$	2	1	0	$\infty$	5/3	5/3
$y$	$\infty$	0	1	1	-1	0	1	$\infty$	-1/9	-1/2

From the coordinates at  $\tau_{-4}$ ,  $\tau_{-8}$ ,  $\tau_{-19}$ ,  $\tau_{-40}$ , and  $\tau_{-88}$ , we see that the relation between  $x$  and  $y$  is

$$a(x+1)y^2 + (-2x^3 + bx^2 + cx + d)y + (2x-3)(x-1)^2 = 0$$

for some rational numbers  $a$ ,  $b$ ,  $c$ , and  $d$ . Then the information at the other CM-points yields

$$a = 1, \quad b = 11, \quad c = -13, \quad d = 2.$$

Setting

$$x_0 = \frac{3-x}{1-x}, \quad y_0 = \frac{4(2x^3 - 11x^2 + 13x - 2 - 2xy - 2y)}{(1-x)^3},$$

we get a Weierstrass model

$$y_0^2 = x_0^6 - 18x_0^4 + 113x_0^2 - 32$$

for  $X$ . Then letting

$$x_1 = x_0^2, \quad y_1 = y_0, \quad x_2 = -32/x_0^2, \quad y_2 = 32y_0/x_0^3,$$

we obtain modular parameterization of two elliptic curves

$$y_1^2 = x_1^3 - 18x_1^2 + 113x_1 - 32, \quad y_2^2 = x_2^3 + 113x_2^2 + 576x_2 + 1024.$$

The minimal models of these two elliptic curves are  $E_{302C1} : Y^2 + XY + Y = X^3 - X^2 + 3$  and  $E_{302A1} : Y^2 + XY + Y = X^3 + X^2 - 230X + 1251$ , respectively, in Cremona's table. The coordinates of the CM-points on the two curves are

	-4	-8	-11	-19	-20	-40	-43	-88	-148	-232
$E_{302A1}$	$2P - Q$	$3P - Q$	$2P$	$4P$	$P$	$3P$	$3P - Q$	$P$	$3P + Q$	$2P - Q$
$E_{302C1}$	$5R$	$R$	$O$	$2R$	$2R$	$O$	$-R$	$-2R$	$5R$	$-5R$

where  $P = (-32, 256)$  generates the torsion subgroup of order 5 and  $Q = (-96, 320)$  generates the free part of  $E_{302A1}(\mathbb{Q})$ , and  $R = (9, 16)$  generates the group of rational points on  $E_{302C1}$ . In the next section, we will address the issue of heights of CM-points on the Jacobians of these elliptic curves.

## 5. HEIGHTS OF CM-DIVISORS ON SHIMURA CURVES

In this section, we will show how to use Borcherds forms to explicitly compute height of CM-points on certain Shimura curves, under Assumption 40. This enables us to verify Zhang's formula [32] for heights of CM-points in certain cases.

**5.1. Zhang's formula.** Zhang's formula [32] holds for Shimura curves over general totally real number fields. For the case of Shimura curves over  $\mathbb{Q}$ , his formula can be described as follows.

Let  $X = X_0^D(N)$  be a Shimura curve. For a newform  $f$  on  $X_0^D(M)$ ,  $M|N$ , let  $a_f(p)$ ,  $p \nmid DN$ , denote the eigenvalue of the Hecke operator  $T_p$  for  $f$ . The Hecke algebra acts on the Jacobian  $\text{Jac}(X)$ . We let  $J_f$  be the maximal abelian subvariety of  $\text{Jac}(X)$  annihilated by all  $T_p - a_f(p)$ ,  $p \nmid DN$ . Then  $\text{Jac}(X)$  is isogenous to  $\prod_{[f]} J_f$ , where the product runs over all Galois conjugacy classes of newforms on  $X_0^D(M)$ ,  $M|N$ .

Assume that  $D \neq 1$  so that  $X$  has no cusps. Define a canonical divisor class

$$(17) \quad \xi = \frac{1}{\text{Vol}(X)} \left( [\Omega_X^1] + \sum_{P \in X} \left( 1 - \frac{1}{e_P} \right) [P] \right)$$

of degree 1 in  $\text{Pic}(X) \otimes \mathbb{Q}$ , where

$$\text{Vol}(X) = \frac{1}{2\pi} \int_X \frac{dx dy}{y^2} = \frac{DN}{6} \prod_{p|D} \left(1 - \frac{1}{p}\right) \prod_{p|N} \left(1 + \frac{1}{p}\right)$$

and  $e_P$  is the cardinality of the stabilizer subgroup of  $P$  in  $X_0^D(N)$  and let  $\iota : X \rightarrow \text{Jac}(X) \otimes \mathbb{Q}$  be defined by  $\iota(P) = P - \xi$ . There exists a positive integer  $n$  such that  $n\iota$  is defined over  $\mathbb{Q}$ .

For a discriminant  $d < 0$  such that a CM-point  $x$  of discriminant  $d$  exists on  $X$ , let  $H_d$  denote the field of definition of  $x$  and set

$$(18) \quad z = \frac{1}{e_x} \sum_{\sigma \in \text{Gal}(H_d/\mathbb{Q}(\sqrt{d}))} \iota(x^\sigma).$$

We have  $z \in \text{Jac}(X)(\mathbb{Q}(\sqrt{d})) \otimes \mathbb{Q}$ . Let  $z_f$  be the  $f$ -isotypical component of  $z$  in  $J \otimes \mathbb{R}$ . Then Zhang's formula gives the Néron-Tate height  $\langle z_f, z_f \rangle$  of  $z_f$  in terms of the derivative of  $L_d(f, s) = L(f, s)L(f, d, s)$  at  $s = 1$ , where  $L(f, d, s)$  denotes the  $L$ -function of the twist of  $f$  by the Kronecker character associated to  $\mathbb{Q}(\sqrt{d})$  over  $\mathbb{Q}$ .

**Theorem D** ([32]). *Under the setting above, we have  $L_d(f, 1) = 0$  and there exists an explicit nonzero constant  $C_f$  depending only on  $f$  such that*

$$\langle z_f, z_f \rangle = C_f \sqrt{|d|} L'_d(f, 1)$$

for all fundamental discriminants  $d$  with  $(d, DN) = 1$ .

For our purpose, we will consider Shimura curves of the form  $X_0^D(N)/W_{D,N}$ . Let  $X = X_0^D(N)/W_{D,N}$  with  $D \neq 1$ . Define  $\xi$  as in (17), but with the formula for  $\text{Vol}(X)$  replaced by

$$\text{Vol}(X) = \frac{DN}{6|W_{D,N}|} \prod_{p|D} \left(1 - \frac{1}{p}\right) \prod_{p|N} \left(1 + \frac{1}{p}\right).$$

Define

$$z = \sum_{\sigma: H_d \hookrightarrow \mathbb{C}} x^\sigma$$

and  $z_f$  analogously as in (18). Note that this time we have  $z \in \text{Jac}(X)(\mathbb{Q}) \otimes \mathbb{Q}$ . Also, for a newform  $f$  on  $X_0^D(N)/W_{D,N}$ , the sign of the functional equation for  $L(f, s)$  is necessarily  $-1$ . Therefore, Zhang's formula becomes

$$\langle z_f, z_f \rangle = C'_f \sqrt{|d|} L(f, d, 1)$$

for all fundamental discriminants  $d$  with  $(d, DN)$  for some nonzero constant  $C'_f$ . In this section, we will discuss how to explicitly compute  $z_f$  and thus verify numerically Zhang's formula for a given Shimura curve. We will consider  $X_0^{142}(1)/W_{142,1}$  and  $X_0^{302}(1)/W_{302,1}$  as our primary examples.

Throughout the rest of this section, for a Shimura curve  $X$  of the form  $X_0^D(N)/W_{D,N}$  and a discriminant  $d < 0$ , we let  $h_d$  be the number of CM-points of discriminant  $d$  on  $X$ . Also, let  $P_d^{\times h_d}$  denote the CM-divisor

$$P_d^{\times h_d} = \sum (\text{CM-points of discriminant } d)$$

of discriminant  $d$  (see Definition 21). If  $h_d = 1$ , we will simply write  $P_d^{\times 1}$  as  $P_d$ . Let  $\text{Div}_{\text{CM}}(X)$  denote the subgroup of  $\text{Div}(X)$  generated by all CM-divisors and  $\text{Div}_{\text{CM}}^0(X)$  be the subgroup of  $\text{Div}_{\text{CM}}(X)$  of degree 0. Set also  $\text{Pic}_{\text{CM}}(X) = \text{Div}_{\text{CM}}(X)/P_{\text{CM}}(X)$



and  $J_{\text{CM}}(X) = \text{Div}_{\text{CM}}^0(X)/P_{\text{CM}}(X)$ , where  $P_{\text{CM}}(X)$  denotes the group of principal divisors contained in  $\text{Div}_{\text{CM}}(X)$ . Since a CM-divisor is defined over  $\mathbb{Q}$ ,  $J_{\text{CM}}(X)$  is a subgroup of  $\text{Jac}(X)(\mathbb{Q})$ .

### 5.2. Example $X_0^{142}(1)/W_{142,1}$ .

**Lemma 44.** *Assume that Assumption 40 holds for the Shimura curve  $X = X_0^{142}(1)/W_{142,1}$ . Then every divisor in  $P_{\text{CM}}(X)$  can be realized as the divisor of a Borchers form. Moreover, the group  $J_{\text{CM}}(X)$  is a free abelian group of rank 1 generated by the divisor class of  $P_{-3} - P_{-4}$ .*

*Proof.* The modular curve  $X_0(284)$  has genus 34. By Lemma 17, when  $n \geq 49$ , we have

$$\dim M_n^1(284) = n - 16.$$

Let

$$t = \frac{\eta(4\tau)^4 \eta(142\tau)^2}{\eta(2\tau)^2 \eta(284\tau)^4} = q^{-35} + 2q^{-33} + q^{-31} + \dots,$$

which is a modular function on  $\Gamma_0(284)$  having only a pole at  $\infty$ . If we can find a positive integer  $n_0$  such that

- (i) for each integer  $k = 0, \dots, 34$ , there exists a linear combination of eta-products in  $M^1(284)$  whose order of pole at  $\infty$  is  $n_0 - k$ , and
- (ii) the space  $M_{n_0}^1(284)$  is spanned by eta-products,

then by multiplying by  $t$  suitably, we can show that all modular forms in  $M^1(284)$  are linear combinations of eta-products. Indeed, our construction shows that  $n_0$  can be as small as 55. It is too complicated to exhibit a basis for  $M_{55}^1(284)$  in terms of eta-products. Here we only remark that the 17 gaps of  $M^1(284)$  are  $1, \dots, 12, 15, 16, 18, 19$  and  $20$ . Moreover, we find that for each non-gap integer  $j$ , there is a modular form  $f_j$  in  $M^1(284) \cap \mathbb{Z}((q))$  with a pole of order  $j$  at  $\infty$  and a leading coefficient 1. Let  $\psi_j$  be the Borchers form associated to  $f_j$ . (Strictly speaking, because the vector-valued modular form constructed using  $f_j$  may not have an integer constant term  $c_0(0)$ , we can only say that a suitable power of  $\psi_j$  is a Borchers form. Here we slightly abuse the terminology.)

Now for a discriminant  $d$  such that there are CM-points of discriminant  $d$  on  $X$ , we let

$$d_0 = \begin{cases} d/4, & \text{if } d \equiv 0 \pmod{4}, \\ d, & \text{if } d \equiv 1 \pmod{4}. \end{cases}$$

When  $|d_0| \geq 21$ ,  $f_{|d_0|}$  exists and we have

$$e_d \text{div } \psi_{|d_0|} = P_d + \sum_{d'} c_{d'} P_{d'},$$

where  $e_d$  is the cardinality of the stabilizer subgroup of CM-points of discriminant  $d$ ,  $d'$  runs over discriminants that are either odd integers with  $|d'| < |d_0|$  or even integers with  $|d'| < 4|d_0|$ . Therefore, recursively, we can show that every  $P_d$  is equivalent to a sum of  $P_{d'}$ , where  $d'$  are either odd discriminants with  $|d'| \leq 21$  or even discriminants with  $|d'| \leq 84$ . Now for the finite list of discriminants with  $|d'| \leq 21$  for the case of odd  $d'$  or  $|d'| \leq 84$  for the case of even  $d'$ , we check case by case that  $P_{d'} \sim n_1 P_{-4} + n_2 P_{-3}$  for some integers  $n_1$  and  $n_2$ . For instance, for  $d' = -8$ , we can construct a modular form in  $M^1(284)$  with Fourier expansion

$$\begin{aligned} & -6q^{-71} - 2q^{-48} + 2q^{-40} - 2q^{-36} + 2q^{-24} - 2q^{-20} + 2q^{-16} - 4q^{-15} - 2q^{-12} \\ & - 2q^{-10} + 2q^{-9} + 2q^{-8} - 6q^{-7} - 2q^{-6} + 2q^{-5} - 2q^{-3} + 2q^{-1} + \dots, \end{aligned}$$

whose corresponding Borchers form has a divisor  $P_{-8} + P_{-4} - 2P_{-3}$ . This gives an algorithmic and recursive way to reduce a CM-divisor to a linear sum of  $P_{-4}$  and  $P_{-3}$ . In particular, we find that  $J_{\text{CM}}(X)$  is generated by  $P_{-4} - P_{-3}$ .

To see that  $J_{\text{CM}}(X)$  has rank 1, i.e.,  $P_{-4} - P_{-3}$  is not a torsion, we recall that in Example 42, we find that an equation for  $X$  is  $E_{142A1} : y^2 + xy + y = x^3 - x^2 - 12x + 15$ . Also, if we let  $Q = (1, 1)$  be the generator of  $E_{142A1}(\mathbb{Q})$ , then  $P_{-3}$  and  $P_{-4}$  are  $-Q$  and  $-2Q$ , respectively. As  $n(-Q - 2Q)$  is not equal to  $O$  for any nonzero integer  $n$ , there cannot exist a modular function on  $X$  with a divisor equal to  $n(P_{-3} - P_{-4})$  for some nonzero integer  $n$ . (Recall that a divisor  $\sum n_i(P_i)$  of an elliptic curve is a principal divisor if and only if  $\sum n_i P_i = O_E$ .) This shows that our construction of Borchers forms does generate all elements in  $P_{\text{CM}}(X)$  and that  $J_{\text{CM}}(X)$  is a free abelian group of rank 1 generated by the class of  $P_{-3} - P_{-4}$ .  $\square$

Of course, the fact that  $J_{\text{CM}}(X)$  is a free abelian group of rank 1 generated by some CM-divisor already follows from Theorems A and C in [32]. Our approach gives an explicit way to compute the heights of CM-divisors on  $J(X)$ . For instance, for a discriminant  $d < 0$ , we let  $n$  be the integer such that  $P_d^{\times h_d} - h_d P_{-4} \sim n(P_{-3} - P_{-4})$ , we have the following data.

$d$	-3	-4	-8	-19	-20	-24	-27	-36	-40	-43
$h_d$	1	1	1	1	1	1	1	1	1	1
$n$	1	0	2	3	4	5	8	7	6	-1
$d$	-72	-75	-83	-91	-100	-107	-116	-120	-131	-147
$h_d$	1	2	3	2	1	3	3	2	5	2
$n$	9	12	14	9	10	13	12	6	21	13
$d$	-148	-152	-171	-179	-180	-187	-196	-200	-216	-219
$h_d$	1	3	4	5	2	2	2	3	3	4
$n$	-2	11	19	19	8	2	11	16	8	15
$d$	-228	-232	-243	-251	-267	-292	-296	-299	$\dots$	-568
$h_d$	2	1	3	7	2	2	5	8	$\dots$	2
$n$	5	-4	3	27	3	15	22	33	$\dots$	20

Now recall that for a prime  $p \neq 2, 71$ , the action of  $T_p$  on  $\text{Div}_{\text{CM}}(X)$  is given by

$$T_p : \frac{1}{e_d} P_d \mapsto \frac{1}{e_{p^2 d}} P_{p^2 d} + \frac{1}{e_d} \left( 1 + \left( \frac{d}{p} \right) \right) P_d,$$

where  $e_d$  is the cardinality of the stabilizer subgroup of a CM-points of discriminant  $d$  on  $X$ . We find  $e_{-3} = 3$ ,  $e_{-4} = 4$ , and

$$\begin{aligned} T_3(P_{-3} - P_{-4}) &= 3P_{-27} + P_{-3} - 4P_{-36} \sim -3(P_{-3} - P_{-4}), \\ T_5(P_{-3} - P_{-4}) &= 3P_{-75}^{\times 2} - 4P_{-100}^{\times 1} - 2P_{-4}^{\times 1} \sim -4(P_{-3} - P_{-4}), \\ T_7(P_{-3} - P_{-4}) &= 3P_{-147}^{\times 2} + 2P_{-3}^{\times 1} - 4P_{-196}^{\times 2} \sim -3(P_{-3} - P_{-4}). \end{aligned}$$

We see that the eigenvalues for  $T_3$ ,  $T_5$ , and  $T_7$  are  $-3$ ,  $-4$ , and  $-3$ , respectively. This agrees with the Cremona's list of Hecke eigenvalues for the cusp form corresponding to  $E_{142A1}$ .

Now we compute heights of CM-divisors for several discriminants. We first determine the canonical divisor class  $\xi$  defined in (17). Since  $X$  has genus 1, the class  $[\Omega_X^1]$  is trivial. It has one elliptic point  $\tau_{-3}$  of order 3, one elliptic point  $\tau_{-4}$  of order 4, and three elliptic points  $\tau_{-8}$ ,  $\tau_{-568}$ , and  $\tau'_{-568}$  of order 2. Its volume is

$$\frac{142}{24} \left(1 - \frac{1}{2}\right) \left(1 - \frac{1}{71}\right) = \frac{35}{12}.$$

Thus,

$$\xi = \frac{1}{35} (8P_{-3}^{\times 1} + 9P_{-4}^{\times 1} + 6P_{-8}^{\times 1} + 6P_{-568}^{\times 2}).$$

According to the table above,

$$35\xi \sim 140P_{-3} - 105P_{-4}.$$

In other words, up to a 35-torsion in  $J_{\text{CM}}(X) \otimes \mathbb{Q}$ , for a CM-divisor  $P_d^{\times h_d}$ , we have

$$P_d^{\times h_d} - h_d\xi \sim (P_d^{\times h_d} - h_dP_{-4}) - 4h_d(P_{-3} - P_{-4}).$$

Let  $n$  be the integer such that the divisor on the right-hand side is equal to  $n(P_{-3} - P_{-4})$ . The table above gives us

$d$	-3	-19	-43	-83	-91	-107	-131
$n$	-3	-1	-5	2	1	1	1
$d$	-179	-187	-219	-251	-267	-299	
$n$	-1	-6	-1	-1	-5	1	

If we numerically approximate  $L(E_{142A1}, d, 1)$ , we find that

$$e_d^2 \sqrt{|d|} L(E_{142A1}, d, 1) \approx n^2 \times 2.619470376720 \dots$$

for the discriminants listed above, agreeing with Zhang's formula. (The evaluation of the  $L$ -values is done by using Magma [5].)

**5.3. Example  $X_0^{302}(1)/W_{302,1}$ .** We next consider  $X = X_0^{302}(1)/W_{302,1}$ . We can show that admissible eta-products span  $M^1(604)$  as before. Also, assuming that Assumption 40 holds for  $X$ , we find that every divisor in  $P_{\text{CM}}(X)$  is the divisor of some Borcherds forms and that

$$J_{\text{CM}}(X) \simeq (\mathbb{Z}/5) \times \mathbb{Z} \times \mathbb{Z},$$

where the torsion subgroup is generated by the divisor class of  $P_{-19} - P_{-20}$ , and the free abelian subgroup is generated by the divisor classes of  $P_{-4} - P_{-20}$  and  $P_{-11} - P_{-20}$ . For those who are interested, we remark that the Borcherds form with a divisor  $5(P_{-19} - P_{-20})$  is coming from the modular form in  $M^1(604)$  with a Fourier expansion

$$\begin{aligned} & -2q^{-144} - 3q^{-128} - 4q^{-124} - 4q^{-100} + 3q^{-84} - 2q^{-80} + 6q^{-76} - 6q^{-72} + 2q^{-68} \\ & - 3q^{-64} + q^{-55} + 26q^{-52} - 4q^{-47} - q^{-44} + 40q^{-41} + 4q^{-40} + 6q^{-39} + q^{-36} \\ & + 5q^{-32} + 6q^{-31} + 19q^{-30} + 12q^{-28} + 19q^{-27} - 12q^{-26} + 4q^{-25} + 28q^{-24} \\ & + 5q^{-23} - 3q^{-21} + 3q^{-20} - q^{-19} + 6q^{-18} - 2q^{-17} + 5q^{-16} - 26q^{-15} \\ & + 7q^{-14} + 14q^{-13} + q^{-11} - 4q^{-10} + q^{-9} + 4q^{-8} + 12q^{-7} + 19q^{-6} - 6q^{-5} \\ & + 3q^{-4} + 28q^{-3} - 6q^{-2} - 5q^{-1} + \dots \end{aligned}$$

Let  $k, m, n$  be the integers,  $0 \leq k \leq 4$ , such that  $P_d^{\times h_d} - h_d P_{-20} \sim k(P_{-19} - P_{-20}) + m(P_{-4} - P_{-20}) + n(P_{-11} - P_{-20})$ . We have the following data.

$d$	-4	-8	-11	-19	-20	-36
$h_d$	1	1	1	1	1	1
$k, m, n$	0, 1, 0	3, 1, 2	0, 0, 1	1, 0, 0	0, 0, 0	2, 0, 2
$d$	-40	-43	-59	-68	-72	-84
$h_d$	1	1	3	2	1	2
$k, m, n$	2, 0, 1	1, 1, 3	1, 0, 1	4, 1, 3	1, -1, -1	0, 1, 2
$d$	-88	-91	-99	-100	-116	-123
$h_d$	1	2	2	1	3	2
$k, m, n$	1, 0, 2	4, 0, -1	3, 1, 0	4, -1, 0	3, 1, 3	1, 2, 5
$d$	-136	-139	-148	-152	-155	-168
$h_d$	2	3	1	3	4	2
$k, m, n$	3, 0, -1	4, 1, 5	2, -1, -3	3, 0, 1	0, 1, 4	3, 1, 1
$d$	-171	-180	-187	-195	$\dots$	-1208
$h_d$	4	2	2	4	$\dots$	6
$k, m, n$	3, 1, 5	4, 1, 5	0, -1, 1	2, 0, 3	$\dots$	4, 0, 4

From the data, we find

$$\begin{aligned}
T_3(P_{-19} - P_{-20}) &= P_{-171}^{\times 4} - P_{-180}^{\times 2} - 2P_{-20}^{\times 1} \sim -(P_{-19} - P_{-20}), \\
T_3(P_{-4} - P_{-20}) &= 4P_{-36}^{\times 1} - P_{-180}^{\times 2} - 2P_{-20}^{\times 1} \\
&\sim 4(P_{-19} - P_{-20}) - (P_{-4} - P_{-20}) + 3(P_{-11} - P_{-20}), \\
T_3(P_{-11} - P_{-20}) &= P_{-99}^{\times 2} + 2P_{-11}^{\times 1} - P_{-180}^{\times 2} - 2P_{-20}^{\times 1} \\
&\sim 4(P_{-19} - P_{-20}) - 3(P_{-11} - P_{-20}).
\end{aligned}$$

Let  $f_A$  and  $f_C$  be the eigenforms on  $X$  corresponding to the newforms on  $X_0(302)$  associated to the elliptic curves  $E_{302A1}$  and  $E_{302C1}$  through the Jacquet-Langlands correspondence. Let  $a_{f_A}(p)$  and  $a_{f_C}(p)$  denote the eigenvalues of  $T_p$  for  $f_A$  and  $f_C$ , respectively. According to Cremona's table, we have  $a_{f_A}(3) = -1$  and  $a_{f_C}(3) = -3$ . Let  $J_{f_A, \text{CM}}$  and  $J_{f_C, \text{CM}}$  be the maximal subgroups of  $J_{\text{CM}}(X)$  annihilated by all  $T_p - a_{f_A}(p)$  and all  $T_p - a_{f_C}(p)$ , respectively. We find  $J_{f_A, \text{CM}}$  is generated by the divisor classes of

$$D_A = 2P_{-4} + 3P_{-11} - 5P_{-20}, \quad D_{A,0} = P_{-19} - P_{-20},$$

where  $D_{A,0}$  is a 5-torsion, and  $J_{f_C, \text{CM}}$  is spanned by

$$D_C = P_{-11} - 2P_{-19} + P_{-20}.$$

We now determine the canonical divisor in Zhang's formula. In Example 43, we find that the unique genus-zero subfield of degree 2 in the function field of  $X$  is generated by a function  $x$  with divisor  $\text{div } x = P_{-43} + P_{-72} - P_{-19} - P_{-88}$ . If we take a standard Weierstrass equation  $y^2 = f(x)$  for  $X$ , then  $\text{div } x/dy = P_{-19} + P_{-88}$ . Thus, a representative for the class  $[\Omega_X^1]$  is  $P_{-19} + P_{-88}$ . The curve  $X$  has one elliptic point  $\tau_{-4}$  of order 4, and

7 elliptic points  $\tau_{-8}$  and  $\tau_{-1208}^{(1)}, \dots, \tau_{-1208}^{(6)}$  of order 2. The volume is

$$\frac{302}{24} \left(1 - \frac{1}{2}\right) \left(1 - \frac{1}{151}\right) = \frac{25}{4}.$$

Therefore, the canonical divisor  $\xi$  in Zhang's formula is

$$\xi = \frac{1}{25} (4P_{-19} + 4P_{-88} + 3P_{-4} + 2P_{-8} + 2P_{-1208}^{\times 6}).$$

Up to a 25-torsion in  $J_{\text{CM}}(X) \otimes \mathbb{Q}$ , we have

$$\xi - P_{-20} \sim \frac{1}{25} (5P_{-4} + 20P_{-11} + 2P_{-19} - 27P_{-20}) = \frac{1}{10}D_A + \frac{27}{25}D_{A,0} + \frac{1}{2}D_C.$$

Now for a discriminant  $d$ , let  $k, m, n$  be the rational numbers such that

$$P_d^{\times h_d} - h_d P_{-20} - h_d \left( \frac{1}{10}D_A + \frac{27}{25}D_{A,0} + \frac{1}{2}D_C \right) = kD_{A,0} + \frac{m}{10}D_A + \frac{n}{2}D_C.$$

We have

$d$	-11	-19	-43	-59	-91	-123	-139	-155	-187	-195
$m, n$	-1, 1	-1, -1	4, 2	-3, -1	-2, -4	8, 2	2, 4	1, 1	-7, 3	-4, 2

Approximating  $L(E_{302A1}, d, 1)$  and  $L(E_{302C1}, d, 1)$  numerically, we find that for the discriminants above, we have

$$\sqrt{|d|}L(E_{302A1}, d, 1) \approx m^2 \times 1.225637922269744563 \dots,$$

and

$$\sqrt{|d|}L(E_{302C1}, d, 1) \approx n^2 \times 4.02890102461114010 \dots$$

**5.4. Example  $X_0^{334}(1)/W_{334,1}$ .** Consider  $X = X_0^{334}(1)/W_{334,1}$ , which has genus 2. Let  $g_1$  and  $g_2$  be eigenforms on  $X$ . The eigenforms on  $\Gamma_0(334)$  corresponding to  $g_1$  and  $g_2$  under the Jacquet-Langlands correspondence are

$$\tilde{g}_1 = q + q^2 + \frac{-3 + \sqrt{5}}{2}q^3 + q^4 + (-2 - \sqrt{5})q^5 + \frac{-3 + \sqrt{5}}{2}q^6 - 3q^7 + q^8 + \dots$$

and its Galois conjugate. We will determine the equation of  $X$  and compute the heights of several CM-divisors.

We can find modular forms  $f_1, \dots, f_4$  in  $M^!(668)$  such that their associated Borchers forms  $\psi_i$  have divisors

$$\begin{aligned} \text{div } \psi_1 &= P_{-19} + P_{-36} - P_{-8} - P_{-27}, \\ \text{div } \psi_2 &= P_{-88} + P_{-100} - P_{-8} - P_{-27}, \\ \text{div } \psi_3 &= P_{-19} + P_{-27} + P_{-232} - P_{-11} - P_{-72} - P_{-100}, \\ \text{div } \psi_4 &= P_{-484}^{\times 3} - P_{-11} - P_{-72} - P_{-100}. \end{aligned}$$

Let  $x$  and  $y$  be the modular functions on  $X$  such that  $\text{div } x = \text{div } \psi_1$ ,  $x(\tau_{-88}) = 1$ ,  $\text{div } y = \text{div } \psi_3$ , and  $y(\tau_{-3}) = 3$ . We find

$d$	-3	-4	-8	-11	-19	-24	-27	-36	-72	-88	-100	-232
$x$	2/3	6	$\infty$	2	0	2	$\infty$	0	2/3	1	1	6
$y$	3	-1/10	2/3	$\infty$	0	-2	0	2	$\infty$	10/3	$\infty$	0

and deduce that the relation between  $x$  and  $y$  is

$$(x-2)(3x-2)(x-1)y^2 + (-2x^3 + 17x^2 - 26x + 8)y - 2x(x-6) = 0.$$

Setting

$$X = x, \quad Y = 2y(3x^3 - 11x^2 + 12x - 4) - 2x^3 + 17x^2 - 26x + 8,$$

we get the Weierstrass equation

$$Y^2 = 4X^6 - 44X^5 + 161X^4 - 292X^3 + 340X^2 - 224X + 64.$$

Furthermore, we find that  $J_{\text{CM}}(X)$  is a free abelian group of rank 2 generated by the divisor classes of

$$D_1 = P_{-8} - P_{-3}, \quad D_2 = P_{-11} - P_{-3}.$$

For a CM-divisor  $P_d^{\times h_d}$ , let  $m$  and  $n$  be the integers such that  $P_d^{\times h_d} - h_d P_{-3} \sim mD_1 + nD_2$ . We have

$d$	-3	-4	-8	-11	-19	-24	-27	-36	-56
$h_d$	1	1	1	1	1	1	1	1	2
$m, n$	0, 0	15, -4	1, 0	0, 1	5, -1	3, 0	2, 1	-2, 2	10, -1
$d$	-72	-75	-84	-88	-99	-100	-107	-115	-116
$h_d$	1	2	2	1	2	1	3	2	3
$m, n$	3, 1	15, -2	9, -1	11, -2	17, -4	-8, 3	0, 2	14, -3	10, 0
$d$	-132	-147	-152	-168	-171	-179	-195	-196	-200
$h_d$	2	2	3	2	4	5	4	2	3
$m, n$	3, 1	11, 0	7, 0	-4, 3	5, 3	10, 2	5, 2	-3, 3	18, -2
$d$	-203	-211	-216	-228	-232	-243	-244	$\dots$	-1336
$h_d$	4	3	3	2	1	3	3	$\dots$	6
$m, n$	8, 1	2, 2	9, 0	12, -1	-12, 5	21, -4	8, 1	$\dots$	12, 4

The action of the Hecke operator  $T_3$  is

$$T_3 D_1 = 2P_{-72} + 2P_{-8} - 3P_{-27} - P_{-3} \sim 2D_1 - D_2,$$

$$T_3 D_2 = P_{-99}^{\times 2} + 2P_{-11} - 3P_{-27} - P_{-3} \sim 11D_1 - 5D_2.$$

Thus, the  $g_1$ -isotypical and  $g_2$ -isotypical components of  $J_{\text{CM}}(X) \otimes \mathbb{R}$  are spanned by

$$\frac{7 + \sqrt{5}}{2} D_1 - D_2, \quad \frac{7 - \sqrt{5}}{2} D_1 - D_2,$$

respectively.

We now verify Zhang's formula for the case of  $X$ . The canonical divisor class in Zhang's formula is the class of

$$\xi = \frac{12}{83} \left( P_{-8} + P_{-27} + \frac{2}{3} P_{-3} + \frac{3}{4} P_{-4} + \frac{1}{2} P_{-8} + \frac{1}{2} P_{-1336}^{\times 6} \right).$$

We find that, up to a 83-torsion in  $J_{\text{CM}}(X) \otimes \mathbb{Q}$ ,

$$\xi - P_{-3} \sim 3P_{-8} - 3P_{-3} = 3D_1.$$

Let

$$E_1 = \frac{1}{\sqrt{5}} \left( \frac{7 + \sqrt{5}}{2} D_1 - D_2 \right), \quad E_2 = -\frac{1}{\sqrt{5}} \left( \frac{7 - \sqrt{5}}{2} D_1 - D_2 \right),$$

and  $s$  and  $t$  be the real numbers in  $\mathbb{Q}(\sqrt{5})$  such that

$$P_d^{\times h_d} - h_d P_{-3} - 3h_d D_1 \sim sE_1 + tE_2.$$

We find

$d$	-3	-11	-19	-107	-115
$s$	-3	$(1 - \sqrt{5})/2$	$(-3 + \sqrt{5})/2$	$-2 - \sqrt{5}$	$(-5 + 3\sqrt{5})/2$
$d$	-179	-195	-203	-211	-251
$s$	$2 - \sqrt{5}$	$-\sqrt{5}$	$-(1 + \sqrt{5})/2$	$-\sqrt{5}$	2

Approximating  $L(g_1, d, 1)$  numerically, we find that

$$e_d^2 \sqrt{|d|} L(g_1, d, 1) \approx s^2 \times 0.2909633434 \dots$$

## Appendices

### APPENDIX APPENDIX A TABLES FOR EQUATIONS OF HYPERELLIPTIC SHIMURA CURVES

We list defining equations of hyperelliptic Shimura curves below.

$X_0^{26}(1)$	$y^2 = -2x^6 + 19x^4 - 24x^2 - 169$
	$w_2(x, y) = (-x, -y),$
	$w_{26}(x, y) = (x, -y).$
$X_0^{35}(1)$	$y^2 = -(x^2 + 7)(7x^6 + 51x^4 + 197x^2 + 1)$
	$w_5(x, y) = (-x, -y),$
	$w_{35}(x, y) = (x, -y).$
$X_0^{38}(1)$	$y^2 = -16x^6 - 59x^4 - 82x^2 - 19$
	$w_2(x, y) = (-x, -y),$
	$w_{38}(x, y) = (x, -y).$
$X_0^{39}(1)$	$y^2 = -(x^4 - x^3 - x^2 + x + 1)(7x^4 - 23x^3 + 5x^2 + 23x + 7)$
	$w_{13}(x, y) = (-\frac{1}{x}, \frac{y}{x^4}),$
	$w_{39}(x, y) = (x, -y).$
$X_0^{51}(1)$	$y^2 = -(x^2 + 3)(243x^6 + 235x^4 - 31x^2 + 1)$
	$w_3(x, y) = (-x, y),$
	$w_{51}(x, y) = (x, -y).$
$X_0^{55}(1)$	$y^2 = -(x^4 - x^3 + x^2 + x + 1)(3x^4 + x^3 - 5x^2 - x + 3)$
	$w_5(x, y) = (-\frac{1}{x}, \frac{y}{x^4}),$
	$w_{55}(x, y) = (x, -y).$
$X_0^{57}(1)$	$y^2 = (3s + 1)(3s^3 + 11s^2 + 17s + 1)$
	$x^2 = -4s^2 + 2s - 1$
	$w_{19}(s, x, y) = (s, x, -y),$ $w_{57}(s, x, y) = (s, -x, y).$
$X_0^{58}(1)$	$y^2 = -2x^6 - 78x^4 - 862x^2 - 1682$
	$w_2(x, y) = (-x, -y),$
	$w_{29}(x, y) = (x, -y).$

TABLE 2. Equations of level one

#### APPENDIX APPENDIX B TABLES OF COORDINATES OF CM-POINTS ON SHIMURA CURVES

In this appendix, we list coordinates of rational CM-points on Shimura curves  $X_0^D(N)/W_{D,N}$  and also the  $x$ -coordinates of CM-points in the equations of  $X_0^D(N)$ . (However, we do not claim that the list of rational CM-points on  $X_0^D(N)/W_{D,N}$  is complete.) In addition, for



$X_0^{62}(1)$	$y^2 = -64x^8 - 99x^6 - 90x^4 - 43x^2 - 8$
	$w_2(x, y) = (-x, y),$ $w_{62}(x, y) = (x, -y).$
$X_0^{69}(1)$	$y^2 = -243x^8 + 1268x^6 - 666x^4 - 2268x^2 - 2187$
	$w_3(x, y) = (-x, y),$ $w_{69}(x, y) = (x, -y).$
$X_0^{74}(1)$	$y^2 = -2x^{10} + 47x^8 - 328x^6 + 946x^4 - 4158x^2$ $-1369$
	$w_2(x, y) = (-x, -y),$ $w_{74}(x, y) = (x, -y).$
$X_0^{82}(1)$	$y^2 = 4s^4 + 4s^3 + s^2 - 2s + 1$ $x^2 = -19s^2 + 18s - 11$
	$w_2(x, y) = (-x, -y),$ $w_{41}(x, y) = (x, -y).$
$X_0^{86}(1)$	$y^2 = -16x^{10} + 245x^8 - 756x^6 - 1506x^4 - 740x^2 - 43$
	$w_2(x, y) = (-x, -y),$ $w_{86}(x, y) = (x, -y).$
$X_0^{87}(1)$	$y^2 = -(x^6 - 7x^4 + 43x^2 + 27)(243x^6 + 523x^4 + 369x^2 + 81)$
	$w_3(x, y) = (-x, y),$ $w_{87}(x, y) = (x, -y).$
$X_0^{93}(1)$	$y^2 = (3s^3 - 7s^2 - 3t - 1)(3s^3 + s^2 - 3s - 9)$ $x^2 = -4s^2 - 6s - 9$
	$w_3(s, x, y) = (s, -x, -y),$ $w_{31}(s, x, y) = (s, x, -y).$
$X_0^{94}(1)$	$y^2 = -8x^8 + 69x^6 - 234x^4 + 381x^2 - 256$
	$w_2(x, y) = (-x, y),$ $w_{94}(x, y) = (x, -y).$

TABLE 3. Equations of level one

some larger  $D$ , we also give several CM-points of degree 2 that are used in our determination of the equations of Shimura curves.

$X_0^{95}(1)$	$y^2 = -(x^8 + x^7 - x^6 - 4x^5 + x^4 + 4x^3 - x^2 - x + 1)$ $\times (7x^8 + 19x^7 + 21x^6 - 13x^4 + 21x^2 - 19x + 7)$
	$w_5(x, y) = (-\frac{1}{x}, \frac{y}{x^6}),$ $w_{95}(x, y) = (x, -y).$
$X_0^{111}(1)$	$y^2 = -(19x^8 - 44x^7 - 16x^6 + 55x^5 + 37x^4 - 55x^3 - 16x^2 + 44x + 19)$ $\times (x^8 - 3x^5 - x^4 + 3x^3 + 1)$
	$w_{37}(x, y) = (-\frac{1}{x}, \frac{y}{x^8}),$ $w_{111}(x, y) = (x, -y).$
$X_0^{119}(1)$	$y^2 = -(7x^{10} - 171x^8 + 758x^6 + 3418x^4 + 4851x^2 + 2401)$ $\times (x^{10} + 3x^8 + 26x^6 + 278x^4 + 373x^2 + 343)$
	$w_7(x, y) = (-x, y),$ $w_{119}(x, y) = (x, -y).$
$X_0^{134}(1)$	$y^2 = -19x^{14} - 8x^{13} + 178x^{12} - 138x^{11} - 625x^{10} + 940x^9 + 383x^8$ $-1486x^7 + 383x^6 + 940x^5 - 625x^4 - 138x^3 + 178x^2$ $-8x - 19$
	$w_2(x, y) = (\frac{1}{x}, \frac{y}{x^7}),$ $w_{134}(x, y) = (x, -y).$
$X_0^{146}(1)$	$y^2 = -11x^{16} + 82x^{15} - 221x^{14} + 214x^{13} + 133x^{12} - 360x^{11} - 170x^{10}$ $+676x^9 - 150x^8 - 676x^7 - 170x^6 + 360x^5 + 133x^4$ $-214x^3 - 221x^2 - 82x - 11$
	$w_{73}(x, y) = (-\frac{1}{x}, \frac{y}{x^8}),$ $w_{146}(x, y) = (x, -y).$
$X_0^{159}(1)$	$y^2 = -(81x^{10} + 207x^8 + 874x^6 - 130x^4 - 11x^2 + 3)$ $\times (2187x^{10} + 8389x^8 + 8878x^6 + 42x^4 - 41x^2 + 1)$
	$w_3(x, y) = (-x, y),$ $w_{159}(x, y) = (x, -y).$

TABLE 4. Equations of level one

CM-point discriminant	$X_0^{26}(1)/W_{26,1}$ $s$	$X_0^{26}(1)/w_{26}$ $x (s = x^2)$
-8	$\infty$	$\infty$
-11	1	$\pm 1$
-19	9	$\pm 3$
-20	5	$\pm\sqrt{5}$
-24	-3	$\pm\sqrt{-3}$
-52	0	0
-67	81/25	$\pm 9/5$

$X_0^{194}(1)$	$y^2 = -19x^{20} - 92x^{19} - 286x^{18} - 592x^{17} - 921x^{16} - 1016x^{15} - 872x^{14}$ $+ 460x^{13} + 1545x^{12} + 1752x^{11} + 34x^{10} - 1752x^9 + 1545x^8$ $- 460x^7 - 872x^6 + 1016x^5 - 921x^4 + 592x^3 - 286x^2$ $+ 92x - 19$
	$w_{97}(x, y) = \left(-\frac{1}{x}, -\frac{y}{x^{10}}\right),$ $w_{194}(x, y) = (x, -y).$
$X_0^{206}(1)$	$y^2 = -8x^{20} + 13x^{18} + 42x^{16} + 331x^{14} + 220x^{12} - 733x^{10}$ $- 6646x^8 - 19883x^6 - 28840x^4 - 18224x^2 - 4096$
	$w_2(x, y) = (-x, y),$ $w_{206}(x, y) = (x, -y).$

TABLE 5. Equations of level one

$X_0^6(11)$	$y^2 = -19x^8 - 166x^7 - 439x^6 - 166x^5 + 612x^4$ $+166x^3 - 439x^2 + 166x - 19$
	$w_2(x, y) = \left(\frac{x+1}{x-1}, -\frac{4y}{(x-1)^4}\right),$ $w_3(x, y) = \left(-\frac{1}{x}, -\frac{y}{x^4}\right),$ $w_{66}(x, y) = (x, -y).$
$X_0^6(17)$	$z^2 = -3x^2 - 16$ $y^2 = 17x^4 - 10x^2 + 9$
	$w_2(x, y, z) = (-x, y, z),$ $w_3(x, y, z) = (x, -y, -z),$ $w_{34}(x, y, z) = (x, -y, z).$
$X_0^6(19)$	$y^2 = -19x^8 + 210x^6 - 625x^4 + 210x^2 - 19$
	$w_2(x, y) = \left(-\frac{1}{x}, -\frac{y}{x^4}\right),$ $w_3(x, y) = \left(\frac{1}{x}, \frac{y}{x^4}\right),$ $w_{114}(x, y) = (x, -y).$
$X_0^6(29)$	$y^2 = -64x^{12} + 813x^{10} - 3066x^8 + 4597x^6 - 12264x^4$ $+13008x^2 - 4096$
	$w_2(x, y) = (-x, y),$ $w_3(x, y) = \left(-\frac{2}{x}, \frac{8y}{x^6}\right),$ $w_{174}(x, y) = (x, -y)$
$X_0^6(31)$	$y^2 = -243x^{12} + 11882x^{10} - 177701x^8 + 803948x^6$ $-1599309x^4 + 962442x^2 - 177147$
	$w_2(x, y) = \left(\frac{3}{x}, -\frac{27y}{x^6}\right),$ $w_3(x, y) = (-x, y),$ $w_{186}(x, y) = (x, -y).$
$X_0^6(37)$	$y^2 = -4096x^{12} - 18480x^{10} - 40200x^8 - 51595x^6$ $-40200x^4 - 18480x^2 - 4096$
	$w_2(x, y) = (-x, y),$ $w_3(x, y) = \left(\frac{1}{x}, \frac{y}{x^6}\right),$ $w_{222}(x, y) = (x, -y).$

TABLE 6. Equations of level greater than one

$X_0^{10}(11)$	$y^2 = -8x^{12} - 35x^{10} + 30x^8 + 277x^6 + 120x^4$ $-560x^2 - 512$
	$w_{10}(x, y) = \left(-\frac{2}{x}, -\frac{8y}{x^6}\right),$ $w_{22}(x, y) = \left(\frac{2}{x}, \frac{8y}{x^6}\right),$ $w_{110}(x, y) = (x, -y).$
$X_0^{10}(13)$	$z^2 = -2x^2 - 25$ $y^2 = 5x^4 - 74x^2 + 325$
	$w_2(x, y, z) = (x, -y, -z),$ $w_5(x, y, z) = (-x, -y, -z),$ $w_{65}(x, y, z) = (x, -y, z).$
$X_0^{10}(19)$	$z^2 = 5x^2 - 32$ $y^2 = -8x^6 + 57x^4 - 40x^2 + 16$
	$w_2(x, y, z) = (-x, y, z),$ $w_5(x, y, z) = (x, -y, -z),$ $w_{38}(x, y, z) = (x, -y, z).$
$X_0^{10}(23)$	$y^2 = -43x^{20} + 318x^{19} - 1071x^{18} + 3014x^{17} - 10540x^{16}$ $+ 28266x^{15} - 72217x^{14} + 81478x^{13} - 62765x^{12} - 68732x^{11}$ $+ 18840x^{10} + 68732x^9 - 62765x^8 - 81478x^7 - 72217x^6$ $- 28266x^5 - 10540x^4 - 3014x^3 - 1071x^2 - 318x - 43$
	$w_2(x, y) = \left(\frac{2x+1}{x-2}, -\frac{5^5 y}{(x-2)^{10}}\right),$ $w_5(x, y) = \left(-\frac{1}{x}, -\frac{y}{x^{10}}\right),$ $w_{230}(x, y) = (x, -y).$
$X_0^{14}(3)$	$z^2 = -9x^2 - 2$ $y^2 = -7x^4 + 22x^2 + 1$
	$w_2(x, y, z) = (-x, y, z),$ $w_3(x, y, z) = (x, -y, -z),$ $w_{14}(x, y, z) = (x, -y, z).$
$X_0^{14}(5)$	$y^2 = -23x^8 - 180x^7 - 358x^6 - 168x^5 - 677x^4$ $+ 168x^3 - 358x^2 + 180x - 23$
	$w_2(x, y) = \left(-\frac{1}{x}, \frac{y}{x^4}\right),$ $w_{14}(x, y) = (x, -y),$ $w_{35}(x, y) = \left(\frac{x+2}{2x-1}, -\frac{25y}{(2x-1)^4}\right).$

TABLE 7. Equations of level greater than one

$X_0^{15}(2)$	$y^2 = -(x^2 + 3)(3x^2 + 4)(x^4 - x^2 + 4)$
	$w_2(x, y) = \left(\frac{2}{x}, -\frac{4y}{x^4}\right),$ $w_3(x, y) = (-x, y),$ $w_5(x, y) = (-x, -y).$
$X_0^{15}(4)$	$z^2 = -3x^2 - 1$ $y^2 = -(4x^2 - x + 1)(4x^2 + x + 1)(5x^2 + 3)$
	$w_4(x, y, z) = (-x, -y, -z),$ $w_3(x, y, z) = (x, y, -z),$ $w_5(x, y) = (x, -y, -z).$
$X_0^{21}(2)$	$z^2 = -x^2 - 3$ $y^2 = -(3x - 1)(3x + 1)(x^2 + 7)(x^2 + 3)$
	$w_2(x, y, z) = (-x, -y, -z),$ $w_3(x, y, z) = (x, y, -z),$ $w_7(x, y) = (x, -y, z).$
$X_0^{22}(3)$	$y^2 = -27x^8 - 308x^6 - 2146x^4 - 308x^2 - 27$
	$w_2(x, y) = \left(-\frac{1}{x}, -\frac{y}{x^4}\right),$ $w_3(x, y) = (-x, y),$ $w_{66}(x, y) = (x, -y).$
$X_0^{22}(5)$	$y^2 = -11x^{12} - 80x^{10} - 240x^8 - 362x^6 - 240x^4 - 80x^2 - 11$
	$w_2(x, y) = \left(\frac{1}{x}, \frac{y}{x^6}\right),$ $w_5(x, y) = \left(-\frac{1}{x}, -\frac{y}{x^6}\right),$ $w_{110}(x, y) = (x, -y).$
$X_0^{26}(3)$	$z^2 = -8x^2 - 3$ $y^2 = x^6 - 2x^4 + 9x^2 + 8$
	$w_2(x, y, z) = (-x, -y, -z),$ $w_3(x, y, z) = (x, -y, -z),$ $w_{26}(x, y, z) = (x, -y, z).$
$X_0^{39}(2)$	$y^2 = -(x^8 + 11x^7 + 52x^6 + 140x^5 + 243x^4 + 280x^3 + 208x^2 + 88x + 16)$ $(7x^4 + 24x^3 + 32x^2 + 24x + 16)(x^4 + 3x^3 + 8x^2 + 12x + 7)$
	$w_4(x, y, z) = (-x, -y, -z),$ $w_3(x, y, z) = (x, y, -z),$ $w_5(x, y) = (x, -y, -z).$

TABLE 8. Equations of level greater than one

CM-point discriminant	$X_0^{38}(1)/W_{38,1}$ $s$	$X_0^{38}(1)/w_{38}$ $x (s = x^2)$
-4	$\infty$	$\infty$
-11	1	$\pm 1$
-19	0	0
-20	-1	$\pm\sqrt{-1}$
-24	-3	$\pm\sqrt{-3}$
-43	9	$\pm 3$
-163	4/81	$\pm 2/9$
-228	-19/9	$\pm\sqrt{-19}/3$
-232	29/81	$\pm\sqrt{29}/9$
-532	-9/49	$\pm 3\sqrt{-1}/7$
-760	-171/16	$\pm 3\sqrt{-19}/4$

TABLE 11. CM-values of  $X_0^{38}(1)$ 

CM-point discriminant	$X_0^{39}(1)/W_{39,1}$ $s$	$X_0^{39}(1)/w_{39}$ $x (s = (2x^2 - 3x - 2)/x)$
-7	$\infty$	$0, \infty$
-15	-1	$(1 \pm \sqrt{5})/2$
-19	-3	$\pm 1$
-24	1	$1 \pm \sqrt{2}$
-28	0	$-1/2, 2$
-60	-5	$(-1 \pm \sqrt{5})/2$
-67	7/3	$-1/3, 3$
-91	-1/3	$(2 \pm \sqrt{13})/3$
-123	1/5	$(4 \pm \sqrt{11})/5$
-163	-57/35	$-5/7, 7/5$
-195	5	$2 \pm \sqrt{5}$
-267	-47/5	$(-8 \pm \sqrt{89})/5$
-312	-17/7	$(1 \pm 5\sqrt{2})/7$
-403	-75/17	$(-6 \pm 5\sqrt{13})/17$

TABLE 12. CM-values of  $X_0^{39}(1)$

CM-point discriminant	$X_0^{51}(1)/W_{51,1}$ $s$	$X_0^{51}(1)/w_{51}$ $x \ (s = x^2)$
-3	$\infty$	$\infty$
-7	1	$\pm 1$
-12	0	0
-24	$-1/3$	$\pm 1/\sqrt{-3}$
-28	$1/9$	$\pm 1/3$
-51	-3	$\pm \sqrt{-3}$
-163	$1/16$	$\pm 1/4$
-187	$-11/9$	$\pm \sqrt{-11}/3$
-267	$-25/3$	$\pm 5/\sqrt{-3}$
-408	$-1/6$	$\pm 1/\sqrt{-6}$

TABLE 13. CM-values of  $X_0^{51}(1)$ 

CM-point discriminant	$X_0^{55}(1)/W_{55,1}$ $s$	$X_0^{55}(1)/w_{55}$ $x \ (s = (x^2 - 1)/x)$
-3	$-3/2$	$1/2, -2$
-12	$\infty$	$0, \infty$
-15	1	$(1 \pm \sqrt{5})/2$
-27	0	$\pm 1$
-60	-1	$(-1 \pm \sqrt{5})/2$
-67	$8/3$	$-1/3, 3$
-88	-2	$-1 \pm \sqrt{2}$
-115	-4	$-2 \pm \sqrt{5}$
-163	$-5/6$	$-3/2, 2/3$
-187	$1/2$	$(1 \pm \sqrt{17})/4$
-235	$-4/11$	$(-2 \pm 5\sqrt{5})/11$
-715	$-4/3$	$(-2 \pm \sqrt{13})/3$

TABLE 14. CM-values of  $X_0^{55}(1)$



CM-point discriminant	$X_0^{57}(1)/W_{57,1}$ $s$	$X_0^{57}(1)/W_{57,1}$ $x$
-4	$\infty$	$\infty$
-7	-1	$\pm\sqrt{-7}$
-16	0	$\pm\sqrt{-1}$
-19	-1/3	$\pm\sqrt{-19}/3$
-24	1	$\pm\sqrt{-3}$
-28	1/3	$\pm\sqrt{-7}/3$
-43	-3	$\pm\sqrt{-43}$
-123	-1/2	$\pm\sqrt{-3}$
-163	-11/6	$\pm\sqrt{-163}/3$
-267	-11	$\pm 13\sqrt{-3}$

TABLE 15. CM-values of  $X_0^{57}(1)$ 

CM-point discriminant	$X_0^{58}(1)/W_{58,1}$ $s$	$X_0^{58}(1)/w_{29}$ $x$ ( $s = -x^2/19$ )
-3	27/19	$\pm 3\sqrt{-3}$
-8	$\infty$	$\infty$
-11	11/19	$\pm\sqrt{-11}$
-19	1	$\pm\sqrt{-19}$
-27	3/19	$\pm\sqrt{-3}$
-40	-5/19	$\pm\sqrt{5}$
-43	43/19	$\pm\sqrt{-43}$
-148	25/19	$\pm 5\sqrt{-1}$
-163	163/475	$\pm\sqrt{-163}/5$
-232	0	0

TABLE 16. CM-values of  $X_0^{58}(1)$

CM-point discriminant	$X_0^{62}(1)/W_{62,1}$ $s$	$X_0^{62}(1)/w_{62}$ $x (s = x^2)$
-4	$\infty$	$\infty$
-8	0	0
-19	1	$\pm 1$
-20	-1	$\pm \sqrt{-1}$
-40	$-1/2$	$\pm 1/\sqrt{-2}$
-67	$1/4$	$\pm 1/2$
-163	25	$\pm 5$
-372	$-4/9$	$\pm 2\sqrt{-1}/3$
-403	$13/4$	$\pm \sqrt{13}/2$

TABLE 17. CM-values of  $X_0^{62}(1)$ 

CM-point discriminant	$X_0^{69}(1)/W_{69,1}$ $s$	$X_0^{69}(1)/w_{69}$ $x (s = x^2)$
-3	$\infty$	$\infty$
-4	1	$\pm 1$
-12	0	0
-24	-3	$\pm \sqrt{-3}$
-75	$9/5$	$\pm 3/\sqrt{5}$
-115	5	$\pm \sqrt{5}$
-123	$-1/3$	$\pm 1/\sqrt{-3}$
-147	$-9/7$	$\pm 3/\sqrt{-7}$
-163	$25/9$	$\pm 5/3$
-483	-27	$\pm 3\sqrt{-3}$

TABLE 18. CM-values of  $X_0^{69}(1)$

CM-point discriminant	$X_0^{74}(1)/W_{74,1}$ $s$	$X_0^{74}(1)/w_{74}$ $x \ (s = (x^2 - 9)/4)$
-8	$\infty$	$\infty$
-19	-2	$\pm 1$
-20	-1	$\pm \sqrt{5}$
-24	-3	$\pm \sqrt{-3}$
-35	$\pm \sqrt{5}$	$\pm 2 \pm \sqrt{5}$
-43	0	$\pm 3$
-51	$-2 \pm \sqrt{-3}$	$\pm 2 \pm \sqrt{-3}$
-52	1	$\pm \sqrt{13}$
-88	-5	$\pm \sqrt{-11}$
-91	$2 \pm \sqrt{13}$	$\pm 2 \pm \sqrt{13}$
-148	$-9/4$	0
-163	10	$\pm 7$

TABLE 19. CM-values of  $X_0^{74}(1)$ 

CM-point discriminant	$X_0^{82}(1)/W_{82,1}$ $s$	$X_0^{82}(1)/w_{41}$ $x$
-3	$1/2$	$\pm 3\sqrt{-3}/2$
-11	0	$\pm \sqrt{-11}$
-19	$\infty$	$\infty$
-24	1	$\pm 2\sqrt{-3}$
-27	-1	$\pm 4\sqrt{-3}$
-52	$1/3$	$\pm 8\sqrt{-1}/3$
-67	$2/3$	$\pm \sqrt{-67}/3$
-88	$-1/2$	$\pm 3\sqrt{-11}/2$
-123	$1/4$	$\pm \sqrt{-123}/4$
-232	$7/13$	$\pm 24\sqrt{-2}/13$

TABLE 20. CM-values of  $X_0^{82}(1)$

CM-point discriminant	$X_0^{86}(1)/W_{86,1}$ $s$	$X_0^{86}(1)/w_{86}$ $x (s = (-x^2 + 9)/4)$
-4	$\infty$	$\infty$
-11	2	$\pm 1$
-24	3	$\pm \sqrt{-3}$
-35	$\pm \sqrt{5}$	$\pm 2 \pm \sqrt{5}$
-40	1	$\pm \sqrt{5}$
-43	9/4	0
-52	5/2	$\pm \sqrt{-1}$
-56	$1 \pm \sqrt{2}$	$\pm \sqrt{5 \pm 4\sqrt{2}}$
-67	0	$\pm 3$
-68	$(5 \pm \sqrt{17})/4$	$\pm \sqrt{4 \pm \sqrt{17}}$
-228	$10 \pm \sqrt{57}$	$\pm \sqrt{19} \pm 2\sqrt{-3}$
-232	-5	$\pm \sqrt{29}$

TABLE 21. CM-values of  $X_0^{86}(1)$ 

CM-point discriminant	$X_0^{87}(1)/W_{87,1}$ $s$	$X_0^{87}(1)/w_{87}$ $x (s = x^2)$
-3	$\infty$	$\infty$
-12	0	0
-15	-3	$\pm \sqrt{-3}$
-19	1	$\pm 1$
-43	9	$\pm 3$
-48	-1	$\pm \sqrt{-1}$
-60	-1/3	$\pm \sqrt{-3}/3$
-147	-9/7	$\pm 3/\sqrt{-7}$
-435	-5/3	$\pm \sqrt{-15}/3$

TABLE 22. CM-values of  $X_0^{87}(1)$

CM-point discriminant	$X_0^{93}(1)/W_{93,1}$ $s$	$X_0^{93}(1)/w_{31}$ $x$
-4	$\infty$	$\infty$
-7	-1	$\pm\sqrt{-7}$
-16	0	$\pm 3\sqrt{-1}$
-19	1	$\pm\sqrt{-19}$
-28	3	$\pm 3\sqrt{-7}$
-51	-3	$\pm 3\sqrt{-3}$
-67	$-1/3$	$\pm\sqrt{-67}/3$
-163	-7	$\pm\sqrt{-163}$
-267	$3/2$	$\pm 3\sqrt{-3}$
-403	$7/3$	$\pm\sqrt{-403}/3$

TABLE 23. CM-values of  $X_0^{93}(1)$ 

CM-point discriminant	$X_0^{94}(1)/W_{94,1}$ $s$	$X_0^{94}(1)/w_{94}$ $x \ (s = 2x^2 - 9)$
-3	-1	$\pm 2$
-4	-9	0
-8	$\infty$	$\infty$
-24	-5	$\pm\sqrt{2}$
-27	-7	$\pm 1$
-51	$\pm\sqrt{17}$	$(\pm 1 \pm \sqrt{17})/2$
-56	$-4 \pm \sqrt{-7}$	$\pm\sqrt{10 \pm 2\sqrt{-7}}/2$
-84	$-6 \pm \sqrt{-7}$	$(\pm\sqrt{-1} \pm \sqrt{7})/2$
-115	$3 \pm 4\sqrt{5}$	$\pm 1 \pm \sqrt{5}$
-148	-17	$\pm 2\sqrt{-1}$
-168	$-3 \pm 2\sqrt{-7}$	$\pm\sqrt{3 \pm \sqrt{-7}}$
-235	$-13/5$	$\pm 4/\sqrt{5}$

TABLE 24. CM-values of  $X_0^{94}(1)$

CM-point discriminant	$X_0^{95}(1)/W_{95,1}$ $s$	$X_0^{95}(1)/w_{95}$ $x \ (s = (x^2 - 1)/2x)$
-7	$\infty$	$0, \infty$
-20	$\pm\sqrt{-1}$	$\pm\sqrt{-1}$
-28	$-3/4$	$-2, 1/2$
-35	$1/2$	$(1 \pm \sqrt{5})/2$
-43	$0$	$\pm 1$
-115	$-1/2$	$(-1 \pm \sqrt{5})/2$
-163	$4/3$	$-1/3, 3$
-235	$-2$	$-2 \pm \sqrt{5}$
-760	$1/7$	$(1 \pm 5\sqrt{2})/7$

TABLE 25. CM-values of  $X_0^{95}(1)$ 

CM-point discriminant	$X_0^{111}(1)/W_{111,1}$ $s$	$X_0^{111}(1)/w_{111}$ $x \ (s = (x^2 - 2x - 1)/(x^2 + x - 1))$
-15	$\infty$	$(-1 \pm \sqrt{5})/2$
-19	$1$	$0$
-24	$0$	$1 \pm \sqrt{2}$
-43	$-2$	$\pm 1$
-51	$-1$	$(1 \pm \sqrt{17})/4$
-52	$\pm\sqrt{-1}$	$(-1 \pm \sqrt{-1})/2, 1 \pm \sqrt{-1}$
-60	$-1/2$	$(1 \pm \sqrt{5})/2$
-148	$(2 \pm 6\sqrt{-1})/5$	$\pm\sqrt{-1}$
-163	$-1/5$	$-1/2, 2$
-267	$-1/3$	$(5 \pm \sqrt{89})/8$
-555	$2$	$-2 \pm \sqrt{5}$

TABLE 26. CM-values of  $X_0^{111}(1)$

CM-point discriminant	$X_0^{119}(1)/W_{119,1}$ $s$	$X_0^{119}(1)/w_{119}$ $x (s = (x^2 - 5)/4)$
-7	$\infty$	$\infty$
-11	-1	$\pm 1$
-28	$-5/4$	0
-51	-2	$\pm\sqrt{-3}$
-56	$\pm\sqrt{2}$	$\pm\sqrt{5 \pm 4\sqrt{2}}$
-63	$(-9 \pm \sqrt{-3})/6$	$\pm\sqrt{-9 \pm 6\sqrt{-3}}/3$
-91	-3	$\pm\sqrt{-7}$
-99	$-1 \pm \sqrt{-3}$	$\pm\sqrt{1 \pm 4\sqrt{-3}}$
-112	$-3/2$	$\pm\sqrt{-1}$
-163	1	$\pm 3$
-232	$(-13 \pm \sqrt{-2})/9$	$(\pm 1 \pm 2\sqrt{-2})/3$
-595	0	$\pm\sqrt{5}$

TABLE 27. CM-values of  $X_0^{119}(1)$ 

CM-point discriminant	$X_0^{134}(1)/W_{134,1}$ $s$	$X_0^{134}(1)/w_{134}$ $x (s = (2x^2 - 5x + 2)/(x^2 - 2x + 1))$
-4	$\infty$	1
-19	2	$0, \infty$
-24	3	$(1 \pm \sqrt{-3})/2$
-35	$\pm\sqrt{5}$	$(\pm 1 \pm \sqrt{5})/2$
-40	1	$(3 \pm \sqrt{5})/2$
-67	$9/4$	-1
-88	5	$(5 \pm \sqrt{-11})/6$
-91	$-5 \pm 2\sqrt{13}$	$(-1 \pm \sqrt{13})/2, (1 \pm \sqrt{13})/6$
-148	$5/2$	$\pm\sqrt{-1}$
-163	0	$1/2, 2$

TABLE 28. CM-values of  $X_0^{134}(1)$

CM-point discriminant	$X_0^{146}(1)/W_{146,1}$ $s$	$X_0^{146}(1)/w_{146}$ $x (s = (x^2 - 1)/x)$
-11	$\infty$	$0, \infty$
-20	1	$(1 \pm \sqrt{5})/2$
-40	-1	$(-1 \pm \sqrt{5})/2$
-43	0	$\pm 1$
-51	$\pm\sqrt{-3}$	$(\pm 1 \pm \sqrt{-3})/2$
-52	3	$(3 \pm \sqrt{13})/2$
-88	2	$1 \pm \sqrt{2}$
-132	$\pm\sqrt{-1}$	$(\pm\sqrt{-1} \pm \sqrt{3})/2$
-232	5	$(5 \pm \sqrt{29})/2$
-292	$\pm 2\sqrt{-1}$	$\pm\sqrt{-1}$

TABLE 29. CM-values of  $X_0^{146}(1)$ 

CM-point discriminant	$X_0^{159}(1)/W_{159,1}$ $s$	$X_0^{159}(1)/w_{159}$ $x (s = (3x^2 + 7)/2)$
-3	$\infty$	$\infty$
-12	$7/2$	0
-19	5	$\pm 1$
-39	$\pm\sqrt{13}$	$\pm\sqrt{-21 \pm 6\sqrt{13}}/3$
-48	2	$\pm\sqrt{-1}$
-57	3	$\pm 1/\sqrt{-3}$
-67	$11/3$	$\pm 1/3$
-75	$19/5$	$\pm 1/\sqrt{5}$
-84	$(5 \pm 2\sqrt{7})/3$	$(\pm 2\sqrt{-1} \pm \sqrt{-7})/3$
-120	$(17 \pm 2\sqrt{10})/3$	$(\pm 2\sqrt{2} \pm \sqrt{5})/3$
-132	$-7 \pm 6\sqrt{3}$	$\pm 2\sqrt{-1} \pm \sqrt{-3}$
-156	$(34 \pm \sqrt{13})/9$	$\pm\sqrt{15 \pm 6\sqrt{13}}/9$
-232	$(91 \pm 2\sqrt{-2})/27$	$\pm\sqrt{-7 \pm 4\sqrt{-2}}/9$
-267	-1	$\pm\sqrt{-3}$
-795	11	$\pm\sqrt{5}$

TABLE 30. CM-values of  $X_0^{159}(1)$



CM-point discriminant	$X_0^{194}(1)/W_{194,1}$ $s$	$X_0^{194}(1)/w_{194}$ $x (s = (x^2 - 1)/x)$
-19	$\infty$	$0, \infty$
-20	1	$(1 \pm \sqrt{5})/2$
-40	-1	$(-1 \pm \sqrt{5})/2$
-51	$\pm\sqrt{-3}$	$(\pm 1 \pm \sqrt{-3})/2$
-52	-3	$(-3 \pm \sqrt{13})/2$
-67	0	$\pm 1$
-123	$(-3 \pm 2\sqrt{-3})/3$	$(-3 \pm \sqrt{-3})/2, (3 \pm \sqrt{-3})/6$
-148	$-1/3$	$(-1 \pm \sqrt{37})/6$
-232	5	$(5 \pm \sqrt{29})/2$
-235	$\pm\sqrt{5}$	$(\pm 3 \pm \sqrt{5})/2$
-388	$\pm 2\sqrt{-1}$	$\pm\sqrt{-1}$

TABLE 31. CM-values of  $X_0^{194}(1)$ 

CM-point discriminant	$X_0^{206}(1)/W_{206,1}$ $s$	$X_0^{206}(1)/w_{206}$ $x (s = 2x^2 + 1)$
-4	1	0
-8	$\infty$	$\infty$
-19	3	$\pm 1$
-52	-1	$\pm\sqrt{-1}$
-56	$\pm\sqrt{-7}$	$\pm\sqrt{-2 \pm 2\sqrt{-7}}/2$
-91	$-2 \pm \sqrt{-7}$	$\pm\sqrt{-6 \pm 2\sqrt{-7}}/2$
-120	$2 \pm \sqrt{-15}$	$\pm\sqrt{2 \pm 2\sqrt{-15}}/2$
-132	$8 \pm \sqrt{33}$	$(\pm\sqrt{3} \pm \sqrt{11})/2$
-163	9	$\pm 2$
-184	$-2 \pm \sqrt{-23}$	$\pm\sqrt{-6 \pm 2\sqrt{-23}}/2$
-232	-3	$\pm\sqrt{-2}$
-235	$19 \pm 8\sqrt{5}$	$\pm 2 \pm \sqrt{5}$
-267	$\pm\sqrt{-3}$	$\pm\sqrt{-2 \pm 2\sqrt{-3}}/2$
-328	$-2 \pm \sqrt{41}$	$\pm\sqrt{-6 \pm 2\sqrt{41}}/2$
-372	$-13 \pm 8\sqrt{3}$	$\pm 2\sqrt{-1} \pm \sqrt{-3}$

TABLE 32. CM-values of  $X_0^{206}(1)$

CM-point discriminant	$X_0^6(11)/W_{6,11}$ $s$	$X_0^6(11)/\langle w_6, w_{11} \rangle$ $t \ (s = (-t^2 + 2)/2)$	$X_0^6(11)/w_{66}$ $x \ (t = (2x^2 + 2)/(x^2 - 2x - 1))$
-19	-1	$\pm 2$	$-1, 0, 1, \infty$
-24	0	$\pm\sqrt{2}$	$-1 \pm \sqrt{2}$
-40	$-1/9$	$\pm 2\sqrt{5}/3$	$-2 \pm \sqrt{5}, (-1 \pm \sqrt{5})/2$
-43	$-1/49$	$\pm 10/7$	$-3, -2, 1/3, 1/2$
-51	$-1/17$	$\pm 6/\sqrt{17}$	$(-3 \pm \sqrt{17})/4, (-3 \pm \sqrt{17})/2$
-52	$-1/25$	$\pm 2\sqrt{13}/5$	$(-3 \pm \sqrt{13})/2, (-2 \pm \sqrt{13})/3$
-84	$1/7$	$\pm 2\sqrt{21}/7$	$(\sqrt{21} \pm \sqrt{-7})/(\sqrt{21} \pm 7)$
-88	$\infty$	$\infty$	$1 \pm \sqrt{2}$
-120	$3/5$	$\pm 2/\sqrt{5}$	$(\sqrt{5} \pm \sqrt{-15})/(5 \pm \sqrt{5})$
-123	$-9/41$	$\pm 10/\sqrt{41}$	$(-5 \pm \sqrt{41})/2, (-5 \pm \sqrt{41})/8$
-132	1	0	$\pm\sqrt{-1}$

TABLE 33. CM-values of  $X_0^6(11)$ 

CM-point discriminant	$X_0^6(17)/W_{6,17}$ $s$	$X_0^6(17)/\langle w_3, w_{34} \rangle$ $x \ (s = x^2)$
-4	0	0
-19	1	$\pm 1$
-43	9	$\pm 3$
-51	$\infty$	$\infty$
-52	-1	$\pm\sqrt{-1}$
-67	$1/4$	$\pm 1/2$
-84	-3	$\pm\sqrt{-3}$
-120	$-1/3$	$\pm 1/\sqrt{-3}$
-123	$1/9$	$\pm 1/3$
-132	3	$\pm\sqrt{3}$
-408	$-16/3$	$\pm 4/\sqrt{-3}$

TABLE 34. CM-values of  $X_0^6(17)$

CM-point discriminant	$X_0^6(19)/W_{6,19}$ $s$	$X_0^6(19)/\langle w_2, w_{57} \rangle$ $t (s = -t^2/4)$	$X_0^6(19)/w_{114}$ $x (t = (x^2 - 1)/x)$
-3	0	0	$\pm 1$
-19	$\infty$	$\infty$	$0, \infty$
-40	$-1/4$	$\pm 1$	$(\pm 1 \pm \sqrt{5})/2$
-51	$3/4$	$\pm \sqrt{-3}$	$(\pm 1 \pm \sqrt{-3})/2$
-52	$-9/4$	$\pm 3$	$(\pm 3 \pm \sqrt{13})/2$
-67	$-9/16$	$\pm 3/2$	$\pm 2, \pm 1/2$
-84	$-3/4$	$\pm \sqrt{3}$	$(\pm \sqrt{3} \pm \sqrt{7})/2$
-88	-1	$\pm 2$	$\pm 1 \pm \sqrt{2}$
-132	$1/4$	$\pm \sqrt{-1}$	$(\pm \sqrt{-1} \pm \sqrt{3})/2$
-148	$-1/36$	$\pm 1/3$	$(\pm 1 \pm \sqrt{37})/6$
-228	1	$\pm 2\sqrt{-1}$	$\pm \sqrt{-1}$

TABLE 35. CM-values of  $X_0^6(19)$ 

CM-point discriminant	$X_0^6(29)/W_{6,29}$ $s$	$X_0^6(29)/\langle w_3, w_{58} \rangle$ $t (s = 18/(t^2 + 18))$	$X_0^6(29)/w_{174}$ $x (t = -12x/(x^2 - 2))$
-4	1	0	$0, \infty$
-24	0	$\infty$	$\pm \sqrt{2}$
-51	$9/17$	$\pm 4$	$(\pm 3 \pm \sqrt{17})/2$
-52	9	$\pm 4\sqrt{-1}$	$\pm \sqrt{-1}, \pm 2\sqrt{-1}$
-67	$1/9$	$\pm 12$	$\pm 1, \pm 2$
-88	$9/25$	$\pm 4\sqrt{2}$	$\pm \sqrt{2}/2, \pm 2\sqrt{2}$
-120	$-3/5$	$\pm 4\sqrt{-3}$	$(\pm \sqrt{-3} \pm \sqrt{5})/2$
-123	$9/41$	$\pm 8$	$(\pm 3 \pm \sqrt{41})/4$
-132	$3/11$	$\pm 4\sqrt{3}$	$(\pm \sqrt{3} \pm \sqrt{11})/2$
-168	$-9/7$	$\pm 4\sqrt{-2}$	$(\pm 3\sqrt{-2} \pm \sqrt{14})/4$
-228	$27/19$	$\pm 4/\sqrt{-3}$	$(\pm 3\sqrt{-3} \pm \sqrt{-19})/2$
-232	$\infty$	$\pm 3\sqrt{-2}$	$\pm \sqrt{-2}$
-267	$81/89$	$\pm 4/3$	$(\pm 9 \pm \sqrt{89})/2$

TABLE 36. CM-values of  $X_0^6(29)$

CM-point discriminant	$X_0^6(31)/W_{6,31}$ $s$	$X_0^6(31)/\langle w_6, w_{31} \rangle$ $t (s = (t^2 + 3)/3)$	$X_0^6(31)/w_{186}$ $x (t = (x^2 - 3)/2x)$
-3	$\infty$	$\infty$	$0, \infty$
-24	0	$\pm\sqrt{-3}$	$\pm\sqrt{-3}$
-43	$4/3$	$\pm 1$	$\pm 3, \pm 1$
-52	$16/3$	$\pm\sqrt{13}$	$\pm 4 \pm \sqrt{13}$
-84	$16/9$	$\pm\sqrt{21}/3$	$(\pm 4\sqrt{3} \pm \sqrt{21})/3$
-88	$16/27$	$\pm\sqrt{-11}/3$	$(\pm 4 \pm \sqrt{-11})/3$
-120	$8/3$	$\pm\sqrt{5}$	$\pm 2\sqrt{2} \pm \sqrt{5}$
-123	$-16/9$	$\pm 5/\sqrt{-3}$	$\pm 3\sqrt{-3}, \pm\sqrt{-3}/3$
-148	$64/27$	$\pm\sqrt{37}/3$	$(\pm 8 \pm \sqrt{37})/3$
-168	$8/9$	$\pm\sqrt{-3}/3$	$(\pm 2\sqrt{6} \pm \sqrt{-3})/3$
-228	$-16/3$	$\pm\sqrt{-19}$	$\pm 4\sqrt{-1} \pm \sqrt{-19}$
-232	$1/3$	$\pm\sqrt{-2}$	$\pm 1 \pm \sqrt{-2}$
-372	1	0	$\pm\sqrt{3}$
-403	$52/27$	$\pm 5/3$	$(\pm 5 \pm 2\sqrt{13})/3$

TABLE 37. CM-values of  $X_0^6(31)$ 

CM-point discriminant	$X_0^6(37)/W_{6,37}$ $s$	$X_0^6(37)/\langle w_6, w_{37} \rangle$ $t (s = t^2/(t^2 + 4))$	$X_0^6(37)/w_{222}$ $x (t = (x^2 - 1)/x)$
-3	0	0	$\pm 1$
-4	1	$\infty$	$0, \infty$
-40	9	$\pm 3/\sqrt{-2}$	$\pm\sqrt{-2}/2, \pm\sqrt{-2}$
-67	$9/25$	$\pm 3/2$	$\pm 1/2, \pm 2$
-84	$-9/7$	$\pm 3\sqrt{-1}/2$	$(\pm\sqrt{7} \pm 3\sqrt{-1})/4$
-120	$-27/5$	$\pm 3\sqrt{-6}/4$	$(\pm\sqrt{10} \pm 3\sqrt{-6})/8$
-123	-3	$\pm\sqrt{-3}$	$(\pm 1 \pm \sqrt{-3})/2$
-132	$27/11$	$\pm 3\sqrt{-3}/2$	$(\pm 3\sqrt{-3} \pm \sqrt{-11})/4$
-148	$\infty$	$\pm 2\sqrt{-1}$	$\pm\sqrt{-1}$
-232	$81/49$	$\pm 9\sqrt{-2}/4$	$\pm\sqrt{-2}/4, \pm 2\sqrt{-2}$
-312	$-3/13$	$\pm\sqrt{-3}/2$	$(\pm\sqrt{13} \pm \sqrt{-3})/4$
-408	$9/17$	$\pm 3\sqrt{2}/2$	$(\pm 3\sqrt{2} \pm \sqrt{34})/4$
-555	$-27/37$	$\pm 3\sqrt{-3}/4$	$(\pm\sqrt{37} \pm 3\sqrt{-3})/8$

TABLE 38. CM-values of  $X_0^6(37)$

CM-point discriminant	$X_0^{10}(11)/W_{10,11}$ $s$	$X_0^{10}(11)/\langle w_{10}, w_{11} \rangle$ $t \ (s = -t^2/2)$	$X_0^{10}(11)/w_{110}$ $x \ (t = (x^2 - 2)/2x)$
-8	$\infty$	$\infty$	$0, \infty$
-35	$7/8$	$\pm\sqrt{-7}/2$	$(\pm 1 \pm \sqrt{-7})/2$
-40	$1$	$\pm\sqrt{-2}$	$\pm\sqrt{-2}$
-43	$-1/8$	$\pm 1/2$	$\pm 2, \pm 1$
-52	$9/8$	$\pm 3\sqrt{-1}/2$	$\pm 2\sqrt{-1}, \pm\sqrt{-1}$
-88	$0$	$0$	$\pm\sqrt{2}$
-120	$3/8$	$\pm\sqrt{-3}/2$	$(\pm\sqrt{5} \pm \sqrt{-3})/2$
-132	$11/8$	$\pm\sqrt{-11}/2$	$(\pm\sqrt{-3} \pm \sqrt{-11})/2$
-187	$-9/8$	$\pm 3/2$	$(\pm 3 \pm \sqrt{17})/2$
-340	$17/8$	$\pm\sqrt{-17}/2$	$(\pm 3\sqrt{-1} \pm \sqrt{-17})/2$
-660	$33/32$	$\pm\sqrt{-33}/4$	$(\pm\sqrt{-1} \pm \sqrt{-33})/4$
-715	$99/104$	$\pm 3\sqrt{-143}/26$	$(\pm 3\sqrt{-143} \pm \sqrt{65})/26$

TABLE 39. CM-values of  $X_0^{10}(11)$ 

CM-point discriminant	$X_0^{10}(13)/W_{10,13}$ $s$	$X_0^{10}(13)/\langle w_2, w_{65} \rangle$ $x \ (s = x^2)$
-3	$1$	$\pm 1$
-35	$5$	$\pm\sqrt{5}$
-40	$\infty$	$\infty$
-43	$9$	$\pm 3$
-52	$0$	$0$
-88	$25/9$	$\pm 5/3$
-120	$-15$	$\pm\sqrt{-15}$
-195	$-5/3$	$\pm\sqrt{-15}/3$
-235	$5/9$	$\pm\sqrt{5}/3$
-312	$-13/3$	$\pm\sqrt{-39}/3$

TABLE 40. CM-values of  $X_0^{10}(13)$

CM-point discriminant	$X_0^{10}(19)/W_{10,19}$ $s$	$X_0^{10}(19)/\langle w_5, w_{38} \rangle$ $x (s = x^2)$
-3	1	$\pm 1$
-8	0	0
-52	$\infty$	$\infty$
-52	-4	$\pm 2\sqrt{-1}$
-67	4/27	$\pm 2/3$
-88	2	$\pm \sqrt{2}$
-148	-1	$\pm \sqrt{-1}$
-228	4/3	$\pm 2/\sqrt{3}$
-280	4/5	$\pm 2/\sqrt{5}$
-532	-1/4	$\pm \sqrt{-1}/2$
-760	32/5	$\pm 4\sqrt{10}/5$

TABLE 41. CM-values of  $X_0^{10}(19)$ 

CM-point discriminant	$X_0^{10}(23)/W_{10,23}$ $s$	$X_0^{10}(23)/\langle w_{10}, w_{23} \rangle$ $t (s = t^2/(t^2 - 5))$	$X_0^{10}(23)/w_{230}$ $x (t = (5x^2 + 5)/(x^2 - 4x - 1))$
-20	0	0	$\pm \sqrt{-1}$
-40	$\infty$	$\pm \sqrt{5}$	$(-1 \pm \sqrt{5})/2$
-43	5/4	$\pm 5$	$-1/2, 0, 2, \infty$
-67	5	$\pm 5/2$	$-3, -1, 1/3, 1$
-88	5/16	$\pm 5/\sqrt{-11}$	$(-1 \pm \sqrt{-11})/6, (1 \pm \sqrt{-11})/2$
-115	1	$\infty$	$2 \pm \sqrt{5}$
-120	5/8	$\pm 5/\sqrt{-3}$	$(2\sqrt{-1} \pm 2\sqrt{-2})/(\sqrt{-1} \pm \sqrt{3})$
-148	5/9	$\pm 5\sqrt{-1}/2$	$(-1 \pm 2\sqrt{-1})/5, 1 \pm 2\sqrt{-1}$
-235	25/16	$\pm 5\sqrt{5}/3$	$(-11 \pm 5\sqrt{5})/2, (1 \pm \sqrt{5})/2$
-520	-5/8	$\pm 5/\sqrt{13}$	$2(1 \pm \sqrt{-2})/(1 \pm \sqrt{13})$

TABLE 42. CM-values of  $X_0^{10}(23)$

CM-point discriminant	$X_0^{14}(3)/W_{14,3}$ $s$	$X_0^{14}(3)/\langle w_3, w_{14} \rangle$ $x (s = x^2)$
-8	0	0
-11	1	$\pm 1$
-35	$-1/7$	$\pm 1/\sqrt{-7}$
-51	$1/9$	$\pm 1/3$
-84	$\infty$	$\infty$
-120	$-1/27$	$\pm \sqrt{-3}/9$
-123	$25/9$	$\pm 5/3$
-168	$-2/9$	$\pm \sqrt{-2}/3$
-228	$-25/27$	$\pm 5/\sqrt{-3}$
-267	$25/1521$	$\pm 5/39$
-312	$49/117$	$\pm 7\sqrt{13}/39$

TABLE 43. CM-values of  $X_0^{14}(3)$ 

CM-point discriminant	$X_0^{14}(5)/W_{14,5}$ $s$	$X_0^{14}(5)/\langle w_5, w_{14} \rangle$ $t (s = t^2)$	$X_0^{14}(5)/w_{14}$ $x (t = (x^2 - x - 1)/(2x^2 + 2))$
-4	$\infty$	$\infty$	$\pm \sqrt{-1}$
-11	1	$\pm 1$	$(-1 \pm \sqrt{-11})/2, (1 \pm \sqrt{-11})/6$
-35	0	0	$(1 \pm \sqrt{5})/2$
-84	-1	$\pm \sqrt{-1}$	$(1 \pm 2\sqrt{-1})(1 \pm \sqrt{21})/10$
-91	-7	$\pm \sqrt{-7}$	$(1 \pm 3\sqrt{13})/(2 \pm 4\sqrt{-7})$
-120	5	$\pm \sqrt{5}$	$(1 \pm 5\sqrt{-3})/(2 \pm 4\sqrt{5})$
-235	$25/81$	$\pm 5/9$	$(-9 \pm \sqrt{5})/2, (9 \pm \sqrt{5})/3$
-280	$5/16$	$\pm \sqrt{5}/4$	$-2 \pm \sqrt{5}$
-340	-25	$\pm 5\sqrt{-1}$	$(1 \pm 10\sqrt{-1})(1 \pm 9\sqrt{5})/202$
-420	$5/9$	$\pm \sqrt{5}/3$	$(3 \pm \sqrt{-35})/(6 \pm 4\sqrt{5})$
-520	$5/81$	$\pm \sqrt{5}/9$	$(9 \pm 5\sqrt{13})/(18 \pm 4\sqrt{5})$
-840	$-35/9$	$\pm \sqrt{-35}/3$	$(3 \pm 11\sqrt{5})/(6 \pm 4\sqrt{-35})$

TABLE 44. CM-values of  $X_0^{14}(5)$

CM-point discriminant	$X_0^{15}(2)/W_{15,2}$ $s$	$X_0^{15}(2)/\langle w_2, w_{15} \rangle$ $t \ (s = t^2)$	$X_0^{15}(2)/\langle w_{15} \rangle$ $x \ (t = (x^2 + 2)/2x)$
-7	1/4	$\pm 1/2$	$(\pm 1 \pm \sqrt{-7})/2$
-12	$\infty$	$\infty$	$\infty, 0$
-15	5/4	$\pm\sqrt{5}/2$	$(\pm\sqrt{5} \pm \sqrt{-3})/2$
-28	9/4	$\pm 3/2$	$\pm 1, \pm 2$
-40	0	0	$\pm\sqrt{-2}$
-48	-1/4	$\pm\sqrt{-1}/2$	$\pm\sqrt{-1}, \pm 2\sqrt{-1}$
-52	1	$\pm 1$	$\pm 1 \pm \sqrt{-1}$
-60	-1/12	$\pm 1/2\sqrt{-3}$	$\pm\sqrt{-3}, \pm 2/\sqrt{-3}$
-88	4	$\pm 2$	$\pm 2 \pm \sqrt{2}$
-120	2	$\pm\sqrt{2}$	$\pm\sqrt{2}$
-132	-1	$\pm\sqrt{-1}$	$\pm\sqrt{-1} \pm \sqrt{-3}$
-148	1/25	$\pm 1/5$	$(\pm 1 \pm 7\sqrt{-1})/5$
-168	2/3	$\pm\sqrt{6}/3$	$(\pm\sqrt{6} \pm 2\sqrt{-3})/3$
-228	-1/9	$\pm\sqrt{-1}/3$	$(\pm\sqrt{-1} \pm \sqrt{-19})/3$
-232	144/121	$\pm 12/11$	$(\pm 12 \pm 7\sqrt{-2})/11$
-240	-25/12	$\pm 5/2\sqrt{-3}$	$\pm 2\sqrt{-3}, \pm\sqrt{-3}/3$
-280	10	$\pm\sqrt{10}$	$\pm 2\sqrt{2} \pm \sqrt{10}$
-312	2/25	$\pm\sqrt{2}/5$	$(\pm\sqrt{2} \pm 4\sqrt{-3})/5$
-340	9/17	$\pm 3/\sqrt{17}$	$(\pm 3 \pm 5\sqrt{-1})/\sqrt{17}$
-372	-31/9	$\pm\sqrt{-31}/3$	$(\pm 7\sqrt{-1} \pm \sqrt{-31})/3$
-408	68/25	$\pm 2\sqrt{17}/5$	$(\pm 3\sqrt{2} \pm 2\sqrt{17})/5$
-420	5/3	$\pm\sqrt{15}/3$	$(\pm\sqrt{15} \pm \sqrt{-3})/3$
-520	-8/121	$\pm 2\sqrt{-2}/11$	$(\pm 2\sqrt{-2} \pm 5\sqrt{-10})/11$
-660	-5/11	$\pm\sqrt{-55}/11$	$(\pm\sqrt{-55} \pm 3\sqrt{-33})/11$
-708	-841/121	$\pm 29\sqrt{-1}/11$	$(\pm 29\sqrt{-1} \pm 19\sqrt{-3})/11$
-760	450/529	$\pm 15\sqrt{2}/23$	$(\pm 15\sqrt{2} \pm 4\sqrt{-38})/23$
-840	40/27	$\pm 2\sqrt{30}/9$	$(\pm 2\sqrt{30} \pm \sqrt{-42})/9$

TABLE 45. CM-values of  $X_0^{15}(2)$



CM-point discriminant	$X_0^{21}(2)/W_{21,2}$ $s$	$X_0^{21}(2)/\langle w_3, w_7 \rangle$ $x \ (s = x^2)$
-4	$\infty$	$\infty$
-7	-7	$\pm\sqrt{-7}$
-15	$-5/3$	$\pm\sqrt{-15}/3$
-16	1	$\pm 1$
-28	$1/9$	$\pm 1/3$
-60	9	$\pm 3$
-84	-3	$\pm\sqrt{-3}$
-100	$1/5$	$\pm 1/\sqrt{5}$
-112	25	$\pm 5$
-120	$-1/3$	$\pm 1/\sqrt{-3}$
-148	$37/9$	$\pm\sqrt{37}/3$
-168	0	0
-228	$-25/3$	$\pm 5/\sqrt{-3}$
-232	-32	$\pm 4\sqrt{-2}$
-280	$-35/9$	$\pm\sqrt{-35}/3$
-312	$-8/3$	$\pm 2\sqrt{-6}/3$
-372	$-3/4$	$\pm\sqrt{-3}/2$
-408	-75	$\pm 5\sqrt{-3}$
-420	21	$\pm\sqrt{21}$
-532	$-19/4$	$\pm\sqrt{-19}/2$
-708	$25/48$	$\pm 5/4\sqrt{-3}$
-840	$-16/3$	$\pm 4/\sqrt{-3}$

TABLE 46. CM-values of  $X_0^{21}(2)$

CM-point discriminant	$X_0^{22}(3)/W_{22,3}$ $s$	$X_0^{22}(3)/\langle w_2, w_{33} \rangle$ $t (s = (t^2 + 1)/t^2)$	$X_0^{22}(3)/w_{66}$ $x (t = (x^2 - 1)/2x)$
-3	1	$\infty$	$0, \infty$
-11	$\infty$	0	$\pm 1$
-20	$5/4$	$\pm 2$	$\pm 2 \pm \sqrt{5}$
-132	0	$\pm \sqrt{-1}$	$\pm \sqrt{-1}$
-168	$27/28$	$\pm 2\sqrt{-7}$	$\pm 2\sqrt{-7} \pm 3\sqrt{-3}$
-267	$169/196$	$\pm 14\sqrt{-3}/9$	$\pm \sqrt{-3}/9, \pm 3\sqrt{-3}$
-312	$25/52$	$\pm 2\sqrt{-39}/9$	$(\pm 2\sqrt{-39} \pm 5\sqrt{-3})/9$
-372	$31/4$	$\pm 2\sqrt{3}/9$	$(\pm 2\sqrt{3} \pm \sqrt{93})/9$
-408	$18/17$	$\pm \sqrt{17}$	$\pm \sqrt{17} \pm 3\sqrt{2}$
-627	$-11/16$	$\pm 4\sqrt{-3}/9$	$(\pm 4\sqrt{-3} \pm \sqrt{33})/9$
-660	$45/44$	$\pm 2\sqrt{11}$	$\pm 2\sqrt{11} \pm 3\sqrt{5}$
-708	$675/676$	$\pm 26\sqrt{-1}$	$\pm 26\sqrt{-1} \pm 15\sqrt{-3}$

TABLE 47. CM-values of  $X_0^{22}(3)$ 

CM-point discriminant	$X_0^{22}(5)/W_{22,5}$ $s$	$X_0^{22}(5)/\langle w_5, w_{22} \rangle$ $t (s = (t^2 + 1)/t^2)$	$X_0^{22}(5)/w_{110}$ $x (t = -2x/(x^2 - 1))$
-4	1	$\infty$	$\pm 1$
-11	$\infty$	0	$0, \infty$
-20	0	$\pm \sqrt{-1}$	$\pm \sqrt{-1}$
-115	$5/4$	$\pm 2$	$(\pm 1 \pm \sqrt{5})/2$
-235	5	$\pm 1/2$	$\pm 2 \pm \sqrt{5}$
-280	$-1/7$	$\pm \sqrt{-14}/4$	$(\pm 2\sqrt{-14} \pm \sqrt{-7})/7$
-520	$5/13$	$\pm \sqrt{-26}/4$	$(\pm 2\sqrt{-26} \pm \sqrt{65})/13$
-660	$-5/4$	$\pm 2\sqrt{-1}/3$	$(\pm 3\sqrt{-1} \pm \sqrt{-5})/2$
-715	$-5/11$	$\pm \sqrt{-11}/4$	$(\pm 4\sqrt{-11} \pm \sqrt{-55})/11$
-760	$-5/76$	$\pm 2\sqrt{-19}/9$	$(\pm 9\sqrt{-19} \pm \sqrt{-95})/38$

TABLE 48. CM-values of  $X_0^{22}(5)$

CM-point discriminant	$X_0^{26}(3)/W_{26,3}$ $s$	$X_0^{26}(3)/\langle w_3, w_{26} \rangle$ $x (s = x^2)$
-8	$\infty$	$\infty$
-11	1	$\pm 1$
-20	-1	$\pm\sqrt{-1}$
-24	0	0
-84	-3	$\pm\sqrt{-3}$
-123	9	$\pm 3$
-132	3	$\pm\sqrt{3}$
-195	$-3/5$	$\pm\sqrt{-15}/5$
-228	$-3/4$	$\pm\sqrt{-3}/2$
-267	$-9/25$	$\pm 3\sqrt{-1}/5$
-312	$-3/8$	$\pm\sqrt{-6}/4$
-372	-12	$\pm 2\sqrt{-3}$
-408	$-3/25$	$\pm\sqrt{-3}/5$
-708	$-27/49$	$\pm 3\sqrt{-3}/7$

TABLE 49. CM-values of  $X_0^{26}(3)$

CM-point discriminant	$X_0^{39}(2)/W_{39,2}$ $s$	$X_0^{39}(2)/\langle w_3, w_{13} \rangle$ $t \ (s = t^2)$	$X_0^{39}(2)/\langle w_{39} \rangle$ $x \ (t = 3(x^2 + 4x + 2)/(x^2 - 2))$
-7	-7	$\pm\sqrt{-7}$	$(-3 \pm \sqrt{-7})/2, (-3 \pm \sqrt{-7})/4$
-15	-15	$\pm\sqrt{-15}$	$(-3 \pm \sqrt{-3})(3 \pm \sqrt{-15})/12$
-24	$\infty$	$\infty$	$\pm\sqrt{2}$
-28	9	$\pm 3$	$\infty, 0, -1, -2$
-52	-9	$\pm 3\sqrt{-1}$	$-1 \pm \sqrt{-1}$
-60	1	$\pm 1$	$-3 \pm \sqrt{5}, (-3 \pm \sqrt{5})/2$
-84	-3	$\pm\sqrt{-3}$	$(3 \pm \sqrt{3})(-3 \pm \sqrt{-3})/6$
-132	-11	$\pm\sqrt{-11}$	$(3 \pm \sqrt{-1})(-3 \pm \sqrt{-11})/10$
-148	-1	$\pm\sqrt{-1}$	$-3 \pm \sqrt{-1}, (-3 \pm \sqrt{-1})/5$
-228	-27	$\pm 3\sqrt{-3}$	$(1 \pm \sqrt{-1})(-1 \pm \sqrt{-3})/2$
-232	-9/2	$\pm 3/\sqrt{-2}$	$-2 \pm \sqrt{-2}, (-2 \pm \sqrt{-2})/3$
-312	0	0	$-2 \pm \sqrt{2}$
-372	-25/3	$\pm 5/\sqrt{-3}$	$(9 \pm \sqrt{-3})(-9 \pm 5\sqrt{-3})/78$
-408	-12	$\pm 2\sqrt{-3}$	$(6 \pm \sqrt{-6})(-3 \pm 2\sqrt{-3})/21$
-520	-10	$\pm\sqrt{-10}$	$(6 \pm \sqrt{-2})(-3 \pm \sqrt{-10})/19$
-708	-59	$\pm\sqrt{-59}$	$(3 \pm 5\sqrt{-1})(-3 \pm \sqrt{-59})/34$
-1092	-1/3	$\pm 1/\sqrt{-3}$	$(9 \pm \sqrt{39})(-9 \pm \sqrt{-3})/42$

TABLE 50. CM-values of  $X_0^{39}(2)$

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